ON THE MATHEMATICAL THEORY OF BOUNDARY LAYER FOR AN UNSTEADY FLOW OF INCOMPRESSIBLE FLUID

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This paper presents the proof of existence of a smooth solution of a system of boundary layer equations for a plane unsteady flow of viscous incompressible fluid in presence of an arbitrary injection and removal of the fluid across the boundary.

It is shown that for such flows, a solution of Prandtl's system always exists for all t near the beginning of flow around the body and, during the interval $0 \leqslant t \leqslant t_1$ along the whole length of this body. A method is given of constructing an approximate solution of the system of Prandtl's equations of the boundary layer theory, and convergence of these approximations is proved. A short resumé of results obtained in this paper is given in [1].

We shall consider a system of boundary layer equations for a plane unsteady flow of a viscous incompressible fluid

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \qquad u_x + v_y = 0 \tag{1}$$

in the region

$$D\,\{0 \leqslant t < t_0,\ 0 \leqslant x < x_0,\ 0 \leqslant y < \infty\} \qquad (t_0 \leqslant \infty,\ x_0 \leqslant \infty) \tag{1.1}$$

with the conditions

$$u\,|_{t=0} = u_0(x, y), \qquad u\,|_{y=0} = 0, \qquad v\,|_{y=0} = v_0(t, x), \qquad u\,|_{x=0} = u_1(t, y) \tag{2}$$

$$\lim_{y\to\infty} u(t, x, y) = U(t, x) \tag{3}$$

Bernoulli's law $U_t + UU_x = -p_x$ connects the functions $p(t, x)$ and $U(t, x)$.

We assume that the density $\rho = 1$.

[2 and 3] give the derivation of these equations. Prandtl's system of equations for a stationary boundary layer is investigated in [4]. Methods which we shall apply in constructing solutions of the problem (1) to (3), can also be used to prove the existence of a solution of the Prandtl's system of equations for a stationary boundary layer.

Physical conditions of the problem demand that $u > 0$ when $y > 0$ and $U(t, x) > 0$.

We shall assume that $u_0 > 0$ and $u_1 > 0$ when $y > 0$; $u_{0y} > 0$ and $u_{1y} > 0$ when $y \geqslant 0$. To prove the existence of a solution to the problem (1) to (3) in D when t_0 or x_0 are restricted in a manner which will be shown later, we shall introduce new independent variables

$$\tau = t, \qquad \xi = x, \qquad \eta = u\,(t,\,x,\,y) \tag{4}$$

and a new unknown function $w = u_y$. We have

$$w_\eta = \frac{u_{yy}}{u_y}, \quad w_{\eta\eta} = \frac{u_{yyy}u_y - u_{yy}^2}{u_y^3}, \quad w_\tau = u_{yt} - \frac{u_{yy}u_t}{u_y}, \quad w_\xi = u_{yx} - \frac{u_{yy}u_x}{u_y}$$

Differentiation of the first equation of (1) with respect to y and subsequent use of both equations of (1) to eliminate v_y and v, leads to the following expression for w

$$L\,(w) \equiv \nu w^2 w_{\eta\eta} - w_\tau - \eta w_\xi + p_x w_\eta = 0 \tag{5}$$

Change of independent variables (4) transforms region D into

$$\Omega\,\{0 \leqslant \tau < t_0,\ 0 \leqslant \xi < x_0,\ 0 \leqslant \eta < U\,(\tau,\,\xi)\}$$

and conditions on the boundary of D become

$$w\,|_{\tau=0} = u_{0y} \equiv w_0\,(\xi,\,\eta), \qquad w\,|_{\xi=0} = u_{1y} \equiv w_1\,(\tau,\,\eta), \qquad w\,|_{\eta=U(\tau,\,\xi)} = 0 \tag{6}$$

$$l\,(w) \equiv \nu w w_\eta - v_0 w - p_x = 0 \text{ when } \eta = 0 \tag{7}$$

on the boundary of Ω. We shall assume u_0 and u_1 to be such that w_0 and w_1 are sufficiently smooth functions on the corresponding boundary of Ω. Also $U\,(\tau,\,\xi) > 0$ for all τ and ξ.

Solution of the problem (5) to (7) will be obtained as a limit of functions w^n as $n \to \infty$, given by

$$L_n\,(w^n) \equiv \nu\,(w^{n-1})^2\, w_{\eta\eta}{}^n - w_\tau{}^n - \eta w_\xi{}^n + p_x w_\eta{}^n = 0 \tag{8}$$

in Ω, where w^n satisfy conditions (6) and

$$l_n\,(w^n) \equiv \nu w^{n-1} w_\eta{}^n - v_0 w^{n-1} - p_x = 0 \tag{9}$$

on the boundary $\eta = 0$ of Ω.

We shall assume that w° is a smooth function satisfying (6) and the condition that $w^\circ > 0$ when $\eta < U\,(\tau,\,\xi)$. Also, we shall assume the existence of such a smooth function $\varphi_0\,(\tau,\,\xi,\,\eta)$ in Ω, that $w^\circ \geqslant \varphi_0\,(0,\,\xi,\,\eta)$, $w_1 \geqslant \varphi_0\,(\tau,\,0,\,\eta)$ and $\varphi_0 > 0$ when $\eta < U\,(\tau,\,\xi)$, while at the same time $\varphi_0 \equiv m_0\,(U\,(\tau,\,\xi) - \eta)^k$ for some $m_0 > 0$ and $k \geqslant 1$, provided that $U\,(\tau,\,\xi) - \eta < \delta_0$ where $\delta_0 > 0$ is a small number.

With the initial assumption that such solution w^n $(n = 1,\,2,\,\ldots)$ of the problem (8), (6) and (9) exists which has continuous derivatives of third order in the closure $\overline{\Omega}$ of Ω, we shall show that w^n converge to the solution of the problem (5) to (7), as $n \to \infty$. This will be followed by a proof of existence of w^n and an approximate method for their construction will be given. We shall also assume that t_0 and x_0 are finite.

Lemma 1. Let a smooth function V be such that $L_n\,(V) \geqslant 0$ in Ω and $l_n\,(V) > 0$ when $\eta = 0$. Let $V \leqslant w^n$ when $\tau = 0$ and $\xi = 0$. Also, let $w^{n-1} > 0$ when $\eta = 0$. Then, $V \leqslant w^n$ everywhere in Ω.

Let a smooth function V_1 be such that $L_n\,(V_1) < 0$ in Ω, $l_n\,(V_1) < 0$ when $\eta = 0$ and let $V_1 \geqslant w^n$ when $\tau = 0$ and $\xi = 0$. Also, let $w^{n-1} > 0$ when $\eta = 0$. Then, $V_1 \geqslant w^n$ everywhere in Ω.

Proof. Let us prove its first part first. Consider the difference $w^n - V = z$. We have

$$L_n(z) = L_n(w^n) - L_n(V) \leqslant 0, \qquad l_n(z) = l_n(w^n) - l_n(V) = \nu\, w^{n-1} z_\eta < 0$$

Previous conditions imply that $z \geqslant 0$ when $\tau = 0$ and when $\xi = 0$. Consider the function $z^1 = ze^{-\tau}$. Obviously, $z^1 \geqslant 0$ when $\tau = 0$ and $\xi = 0$, and $z^1 < 0$ when $\eta = 0$. From this it follows that z^1 cannot assume a negative minimum when $\eta = 0$, since at this point we have $z^1 \geqslant 0$. On the points belonging to $\overline{\Omega}$ we have

$$L_n(z) = (L_n(z^1) - z^1)\, e^\tau \leqslant 0 \tag{10}$$

from which it follows that z^1 cannot assume a negative minimum on the internal point of Ω, nor when $\xi = x_0$ or $\tau = t_0$, since at such points $z^1_\eta = 0$, $z^1_\xi \leqslant 0$, $z^1_\tau \leqslant 0$ and $z^1_{\eta\eta} \geqslant 0$, from which it follows that $L_n(z^1) - z^1 > 0$. Nor can z^1 assume a negative minimum on the boundary $\eta = U(\tau, \xi)$ since we have, on this surface, $w^{n-1} = 0$ while at the minimum point z^1 we have, provided it can be reached, that when $\eta = U(\tau, \xi)$, $-z^1_\tau - \eta z^1_\xi + p_x z^1_\eta = 0$, hence $L_n(z^1) - z^1 > 0$. The latter follows from the fact that the vector $(-1, -\eta, p_x)$ either lies on the plane tangent to the surface $\eta = U(\tau, \xi)$ or, by Bernoulli's law, it is orthogonal to the normal vector

$$U_t + \eta U_\xi + p_x = U_\tau + UU_\xi + p_x = 0$$

Hence, $z^1 \geqslant 0$ in Ω and $w^n \geqslant V$ in Ω. Remaining part of Lemma 1 is proved in the analogous manner.

Lemma 2. There exists a constant $\tau_0 > 0$ such, that for all n and $\tau \leqslant \tau_0$, the inequalities $H_1(\tau, \xi, \eta) \geqslant w^n \geqslant h_1(\tau, \xi, \eta)$, where H_1 is continuous in $\overline{\Omega}$ and the function $h_1(\tau, \xi, \eta)$ is positive for $\eta < U(\tau, \xi)$, $\tau \leqslant \tau_0$ and continuous in Ω, are fulfilled in the region $\overline{\Omega}$.

Proof. Let us construct the functions V and V_1 satisfying the conditions of Lemma 1. We shall define a twice continuously differentiable function $\psi(\tau, \xi, \eta)$ as follows. Let

$$\psi \equiv \varkappa(\alpha_1\eta) \text{ when } \eta < \delta_1, \quad 0 < \delta_1 < {}^1/_2 \min U(\tau, \xi)$$

$$\varkappa(s) = e^s \text{ when } 0 \leqslant s \leqslant 1; \qquad 1 \leqslant \varkappa(s) \leqslant 3 \text{ при } s \geqslant 1$$

$$\psi = (U(\tau, \xi) - \eta)^k \text{ when } U - \eta < \delta_0; \quad 0 < a_0 \leqslant \psi \leqslant 4 \text{ when } \delta_1 < \eta < U - \delta_0$$

where a_0 is a small number. Let the functions V and V_1 be

$$V = m\psi e^{-\alpha\tau} \qquad (m, \alpha_1, \alpha = \text{const} > 0)$$
$$V_1 = M(C - e^{\beta_1\eta})\, e^{\beta\tau} \qquad (\beta_1, \beta, C, M = \text{const} > 0)$$

We shall show that the constants entering V and V_1 can, together with a number $\tau_0 > 0$, be chosen independent of n and in such a manner, that $V \leqslant w^{n-1} \leqslant V_1$ for $\tau < \tau_0$, implies that $V \leqslant w^n \leqslant V_1$ for $\tau \leqslant \tau_0$. Let us consider $l_n(V)$ and $l_n(V_1)$. When $e^{-\alpha\tau} \geqslant 1/2$, we have

$$l_n(V) = \nu w^{n-1} m\psi_\eta e^{-\alpha\tau} - v_0 w^{n-1} - p_x \geqslant me^{-\alpha\tau}[\nu m\alpha_1 e^{-\alpha\tau} - v_0] - p_x > 0$$
$$l_n(V_1) = -\nu w^{n-1} M\beta_1 e^{\beta\tau} - v_0 w^{n-1} - p_x \leqslant me^{-\alpha\tau}(-\nu\beta_1 Me^{\beta\tau} - v_0) - p_x > 0$$

provided $\alpha_1 > 0$ and $\beta_1 > 0$ are sufficiently large.

Constants m, C, and M shall be chosen accordingly from

$$\varphi_0(\tau, \xi, \eta) \geqslant m\psi(\tau, \xi, \eta), \; C - e^{\beta_1\eta} \geqslant 1, \; M \geqslant \max\{w_0, w_1\}$$

Let us now choose $\beta > 0$ such that $L_n(V_1) < 0$ in $\overline{\Omega}$. Taking into account the fact that $w^{n-1} \geqslant V = m\psi e^{-\alpha\tau}$, we have,

$$L_n(V_1) = -\nu(w^{n-1})^2 M\beta_1^2 e^{\beta_1\eta} e^{\beta\tau} - M(C - e^{\beta_1\eta})\beta e^{\beta\tau} - p_x M\beta_1 e^{\beta_1\eta} e^{\beta\tau} \leqslant$$
$$\leqslant -e^{\beta\tau}[\nu(m\psi e^{-\alpha\tau})^2 M\beta_1^2 e^{\beta_1\eta} + M\beta + p_x M\beta_1 e^{\beta_1\eta}] < 0$$

provided that $\beta > 0$ was chosen sufficiently large.

Let us now compute $L_n(V)$. We have

$$L_n(V) = \nu(w^{n-1})^2 m\psi_{\eta\eta} e^{-\alpha\tau} + \alpha m\psi e^{-\alpha\tau} - m\psi_\tau e^{-\alpha\tau} - \eta m\psi_\xi e^{-\alpha\tau} + p_x m\psi_\eta e^{-\alpha\tau}$$

Since $0 \leqslant w^{n-1} \leqslant M(C - e^{\beta_1\eta})e^{\beta\tau}$, the constant $\alpha > 0$ can be chosen independent of n and sufficiently large to ensure that $L_n(V) > 0$ in Ω when $\eta < U(\tau, \xi) - \delta_0$, as $\psi \geqslant \min\{a_0, 1\}$. In the region $\eta \geqslant U(\tau, \xi) - \delta_0$ where $\psi = (U - \eta)^k$, we have

$$L_n(V) = me^{-\alpha\tau}[\nu(w^{n-1})^2 k(k-1)(U-\eta)^{k-2} - k(U-\eta)^{k-1}U_\tau + \alpha(U-\eta)^k -$$
$$- \eta k(U-\eta)^{k-1}U_\xi - p_x k(U-\eta)^{k-1}]$$

From Bernoulli's law it follows that $U_\tau + \eta U_\xi + p_x = -(U-\eta)U_\xi$. Therefore

$$L_n(V) \geqslant me^{-\alpha\tau}[k(U-\eta)^k U_\xi + \alpha(U-\eta)^k] \geqslant 0$$

provided $\alpha > 0$ is sufficiently large. Consequently, conditions of Lemma 1 are fulfilled for V and V_1 in Ω, if $\tau \leqslant \tau_0$ and τ_0 is such that $e^{-\alpha\tau_0} = \frac{1}{2}$. Values of α and τ_0 depend only on the parameters of the problem (5) to (7). Therefore, if $V_1 \geqslant w^{n-1} \geqslant V$ when $\tau \leqslant \tau_0$, then all the conditions of Lemma 1 are fulfilled for V and V_1 and $V_1 \geqslant w^n \geqslant V$ for $\tau \leqslant \tau_0$. Since it can be assumed that these inequalities are also fulfilled for w^0 at any value of n and $\tau \leqslant \tau_0$, we have $V \leqslant w^n < V_1$, which completes the proof.

Lemma 3. There exists a constant $\xi_0 > 0$ such, that for all n and $\xi \leqslant \xi_0$ the inequalities $H_2(\tau, \xi, \eta) \geqslant w^n \geqslant h_2(\tau, \xi, \eta)$, where H_2 is continuous in $\overline{\Omega}$ and a continuous function $h_2(\tau, \xi, \eta)$ is positive for $\eta < U(\tau, \xi)$, $\xi \leqslant \xi_0$, are fulfilled in Ω.

Proof. We shall construct functions V and V_1 satisfying the conditions of Lemma 1. Let $\psi(\tau, \xi, \eta)$ be a function constructed in the proof of Lemma 2, and let $\varphi(s)$ be a function twice differentiable when $s \geqslant 0$, equal to $3 - e^s$ when $0 \leqslant s \leqslant \frac{1}{2}$ and such, that $1 \leqslant \varphi(s) \leqslant 3$ for all s, $|\varphi'| \leqslant 3$, $|\varphi''| \leqslant 3$.

Let also $V = m\psi e^{-\alpha\xi}$ and $V_1 = M\varphi(\beta_1\eta)e^{\beta\xi}$. We shall show that positive constants m, M, α_1, α, β_1, β and a number $\xi_0 > 0$ can be chosen independent of n and such, that when $V_1 \geqslant w^{n-1} \geqslant V$ for $\xi \leqslant \xi_0$, we also have $V_1 \geqslant w^n \geqslant V$ for $\xi \leqslant \xi_0$. Let us consider $l_n(V)$. We have

$$l_n(V) = \nu w^{n-1} m\alpha_1 e^{-\alpha\xi} - v_0 w^{n-1} - p_x = w^{n-1}(\nu m\alpha_1 e^{-\alpha\xi} - v_0) - p_x \geqslant$$
$$\geqslant me^{-\alpha\xi}(\nu m\alpha_1 e^{-\alpha\xi} - v_0) - p_x > 0$$

for sufficiently large α_1 and under the assumption that $e^{-\alpha\xi} \geqslant \frac{1}{2}$. Further

$$l_n(V_1) = -\nu w^{n-1} M\beta_1 e^{\beta\xi} - w^{n-1}v_0 - p_x \leqslant me^{-\alpha\xi}(-\nu M\beta_1 e^{\beta\xi} - v_0) - p_x < 0,$$

if β_1 is sufficiently large and $e^{-\alpha\xi} \geqslant \frac{1}{2}$. Let us now choose $\beta > 0$ so as to fulfill the inequality $L_n(V_1) < 0$. We have

$$L_n(V_1) = \nu(w^{n-1})^2 M\beta_1^2\varphi'' e^{\beta\xi} - \eta M\varphi\beta e^{\beta\xi} + p_x M\beta_1\varphi' e^{\beta\xi} \tag{11}$$

It can easily be seen that $\varphi'' \leqslant -1$ when $\beta_1\eta \leqslant 1/2$. By the previous assumption

$w^{n-1} \geqslant m\psi e^{-\alpha\xi}$, where ψ is fixed, while m is found from the condition that $m\psi \leqslant \varphi_0$, and $e^{-\alpha\xi} \geqslant \frac{1}{2}$ when $\xi \leqslant \xi_0$ by virtue of the choice of ξ_0. Consequently β_1 can be chosen large enough to ensure that $L_n(V_1) < 0$ when $\beta_1\eta \leqslant \frac{1}{2}$. Further, we shall choose $\beta > 0$ large enough to ensure that $L_n(V_1) < 0$ also when $\beta_1\eta \geqslant \frac{1}{2}$. This is permissible, since the second term of (11) can be made arbitrarily large for sufficiently large β, provided $\eta \geqslant \frac{1}{2}\beta_1$. Suitable choice of M leads to the condition $V_1 \geqslant w^n$ being fulfilled when $\tau = 0$ and $\xi = 0$. By Lemma 1 we have $w^n \leqslant V_1$ everywhere in Ω when $\xi \leqslant \xi_0$. Let us now consider $L_n(V)$. We have

$$L_n(V) = \nu (w^{n-1})^2 \psi_{\eta\eta} m e^{-\alpha\xi} - m\psi_\tau e^{-\alpha\xi} + \eta m\psi\alpha e^{-\alpha\xi} - \eta m\psi_\xi e^{-\alpha\xi} + p_x\psi_\eta m e^{-\alpha\xi}$$

Let $\alpha_1\eta \leqslant 1$ and $e^{-\alpha\xi} \geqslant \frac{1}{2}$. Then

$$L_n(V) \geqslant \nu m^3\alpha_1^2 e^{3\alpha_1\eta} e^{-3\alpha\xi} + p_x\alpha_1 e^{\alpha_1\eta} e^{-\alpha\xi} m > 0$$

for sufficiently large α_1, since by the previous assumption, $w^{n-1} \geqslant m\psi e^{-\alpha\xi}$.

Let $1/\alpha_1 < \eta < U - \delta_0$. Then $L_n(V) > 0$, since by the previous assumption $0 \leqslant w^{n-1} \leqslant M\varphi(\beta_1\eta)\, e^{\beta\xi}$ $\eta m\psi\alpha e^{-\alpha\xi}$ can be made arbitrarily large provided α is sufficiently large, otherwise when $1/\alpha_1 < \eta < U(\tau, \xi) - \delta_0$ function $\psi \geqslant a_0 > 0$, while the remaining terms in the expression for $L_n(V)$ are uniformly bounded in n. When $U(\tau, \xi) - \eta < \delta_0$, we have

$$L_n(V) = me^{-\alpha\xi}\,[\nu(w^{n-1})^2 k(k-1)(U-\eta)^{k-2} - k(U-\eta)^{k-1}U_\tau - \\ - \eta k(U-\eta)^{k-1}U_\xi - p_x k(U-\eta)^{k-1} + \alpha\eta(U-\eta)^k]$$

Using Bernoulli's law in the manner employed in the proof of Lemma 2 we obtain, that $L_n(V) > 0$ for $U - \eta < \delta_0$ if α is sufficiently large. From this it follows that when $0 \leqslant \xi \leqslant \xi_0$ and ξ_0 is chosen so that $e^{-\alpha\xi_0} = \frac{1}{2}$, then $L_n(V) > 0$ in Ω. Since m was chosen so that the inequality $V \leqslant w^n$ is true for $\tau = 0$ and $\xi = 0$ we have, by Lemma 1, $w^n \geqslant m\psi e^{-\alpha\xi}$ for $\xi \leqslant \xi_0$ and for all τ. This proves Lemma 3, since we can safely assume that $V \leqslant w^\circ \leqslant V_1$.

In the following we shall only consider such regions of Ω in which either $t_0 \leqslant T_0$ or $x_0 \leqslant \xi_0$.

To obtain the estimates of first and second order derivatives of w^n, we shall prove the Lemmas 4 and 5. We shall introduce in (8), (6) and (9) a new function $W^n = w^n e^{\alpha\eta}$, where $\alpha > 0$ is a constant which will be specified later. We have

$$L_n(w^n) = \nu (w^{n-1})^2 W_{\eta\eta}^n - W_\tau^n - \eta W_\xi^n + [p_x - 2\nu(w^{n-1})^2\alpha]\, W_\eta^n + \\ + [\alpha^2\nu(w^{n-1})^2 - p_x\alpha]\, W^n = 0$$

$$l_n(w^n) = \nu W^{n-1}W_\eta^n - \alpha\nu W^{n-1}W^n - W^{n-1}v_0 - p_x = 0 \quad \text{when } \eta = 0$$

Putting

$$L_n^\circ(W) \equiv \nu(w^{n-1})^2 W_{\eta\eta} - W_\tau - \eta W_\xi + A^n W_\eta, \qquad A^n \equiv [p_x - 2\nu(w^{n-1})^2\alpha]$$

we obtain

$$L_n^\circ(W^n) + B^n W^n = 0, \qquad B^n \equiv [\alpha^2\nu(w^{n-1})^2 - \alpha p_x]$$

Let us now consider the function

$$\Phi_n = (W_\tau^n)^2 + (W_\xi^n)^2 + W_\eta^n(W_\eta^n - 2H^n) + K_0 + K_1\eta$$

$$\left(H^n \equiv \frac{1}{\nu} v_0 + \frac{p_x}{\nu W^{n-1}} + \alpha W^n \chi(\eta)\right)$$

We shall assume that H^n is defined in Ω, while v_0 and p_x are additionally defined for $\eta > 0$ so, that they are equal to zero when $\eta > \delta_2$ where $\delta_2 = ½ \min U(\tau, \xi)$, are independent of η when $\eta < ½\, \delta_2$ and are sufficiently smooth for all η, and that $\chi(\eta) = 1$ when $\eta \leqslant ½\, \delta_2$ and $\chi(\eta) = 0$ when $\eta \geqslant \delta_2$. Obviously $W^n_\eta = H^n$ when $\eta = 0$.

Lemma 4. Constants K_0 and K_1 can be chosen independent of n and such, that

$$\frac{\partial\Phi_n}{\partial\eta} \geqslant \alpha\Phi_n - \frac{\alpha}{2}\Phi_{n-1} \tag{12}$$

when $\eta = 0$, and

$$L_n^\circ(\Phi_n) + R^n\Phi_n \geqslant 0 \tag{13}$$

in Ω where R^n is a function of w^{n-1} and its first and second order derivatives.

Proof. Let us consider $\partial\Phi_n / \partial\eta$ when $\eta = 0$. We have

$$\frac{\partial\Phi_n}{\partial\eta} = 2W_\tau^nW_{\tau\eta}^n + 2W_\xi^nW_{\xi\eta}^n + W_{\eta\eta}^n(W_\eta^n - 2H^n) + W_\eta^n(W_{\eta\eta}^n - 2H_\eta^n) + K_1$$

Using the boundary condition $W_\eta^n - H^n = 0$ when $\eta = 0$, we obtain

$$\frac{\partial\Phi_n}{\partial\eta} = 2W_\tau^nH_\tau^n + 2W_\xi^nH_\xi^n - 2H^nH_\eta^n + K_1$$

By Lemmas 2 and 3, the inequalities $W^n \geqslant h_0 > 0$ hold when $\eta = 0$, and we have

$$H^nH_\eta^n = \left(\frac{1}{\nu}v_0 + \frac{p_x}{\nu W^{n-1}} + \alpha W^n\chi(\eta)\right)\left(-\frac{p_xW_\eta^{n-1}}{\nu(W^{n-1})^2} + \alpha\chi W_\eta^n\right)$$

Let us use the conditions $W^n_\eta - H^n = 0$ to define W^n and W^{n-1}. We shall find, that $H^nH^n_\eta$ depends only on W^n, W^{n-1} and W^{n-2} and is therefore uniformly bounded in n. Consequently, $|2H^nH_\eta^n| \leqslant K_2$ and K_2 is independent of n. Estimating $W_\tau^nH_\tau^n$ and $W_\xi^n H_\xi^n$ we obtain, for $\eta = 0$,

$$W_\tau^nH_\tau^n = W_\tau^n\left[\frac{1}{\nu}v_{0\tau} + \frac{p_{x\tau}}{\nu W^{n-1}} - \frac{p_xW_\tau^{n-1}}{\nu(W^{n-1})^2} + \alpha W_\tau^n\chi(\eta)\right] \geqslant$$

$$\geqslant \alpha(W_\tau^n)^2 - \frac{1}{\alpha}\left[\frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}}\right]^2 - \frac{1}{\alpha}\left[\frac{p_x}{\nu(W^{n-1})^2}\right]^2(W_\tau^{n-1})^2 - \frac{\alpha}{2}(W_\tau^n)^2$$

Choosing $\alpha > 0$ independent of n and such that

$$\frac{1}{\alpha}\left[\frac{p_x}{\nu(W^{n-1})^2}\right]^2 \leqslant \frac{\alpha}{4}$$

we obtain

$$W_\tau^nH_\tau^n \geqslant \frac{\alpha}{2}(W_\tau^n)^2 - \frac{\alpha}{4}(W_\tau^{n-1})^2 - K_3, \qquad K_3 \geqslant \max\frac{1}{\alpha}\left[\frac{v_{0\tau}}{\nu} + \frac{p_{x\tau}}{\nu W^{n-1}}\right]^2$$

Here K_3 is independent of n. Analogously we have

$$W_\xi^nH_\xi^n \geqslant \frac{\alpha}{2}(W_\xi^n)^2 - \frac{\alpha}{4}(W_\xi^{n-1})^2 - K_4, \qquad K_4 \geqslant \max\frac{1}{\alpha}\left[\frac{v_{0\xi}}{\nu} + \frac{p_{x\xi}}{\nu W^{n-1}}\right]^2$$

and, for $\eta = 0$,

$$\frac{\partial\Phi_n}{\partial\eta} \geqslant \alpha[(W_\tau^n)^2 + (W_\xi^n)^2] - \frac{\alpha}{2}[(W_\tau^{n-1})^2 + (W_\xi^{n-1})^2] - K_5 + K_1$$

$$(K_5 = K_2 + 2K_3 + 2K_4)$$

Since $W^n_\eta - H^n = 0$ implies that $W^n_\eta (W^n_\eta - 2H^n)$ is uniformly bounded in n when $\eta = 0$, we can write that

$$\frac{\partial \Phi_n}{\partial \eta} \geqslant \alpha \Phi_n - \frac{\alpha}{2} \Phi_{n-1} - K_6 + K_1$$

Here K_6 is a constant independent of n. Let us choose $K_1 > K_6$. Then, we can easily see that when $\eta = 0$ $\partial\Phi_n / \partial\eta \geqslant \alpha\Phi_n - {}^1/_2\alpha\Phi_{n-1}$, which is precisely what was required to prove. Choosing a suitable value for K_0, we can also assume that $\Phi_n \geqslant 1$ in Ω.

Let us now consider $L_n^\circ(\Phi_n)$. When $\eta \geqslant \delta_2$, $H^n \equiv 0$. Therefore, for such η

$$\Phi_n = \Phi_n^* \equiv (W_\tau^{\ n})^2 + (W_\xi^{\ n})^2 + (W_\eta^{\ n})^2 + K_0 + K_1\eta$$

Applying to $L_n^\circ(W^n) + B^n W^n = 0$ the operator

$$2W_\tau^{\ n} \frac{\partial}{\partial \tau} + 2W_\xi^{\ n} \frac{\partial}{\partial \xi} + 2W_\eta^{\ n} \frac{\partial}{\partial \eta}$$

we obtain

$$\begin{gathered}\nu (w^{n-1})^2 \Phi^*_{n\eta\eta} - \Phi^*_{n\tau} - \eta\Phi^*_{n\xi} + A^n\Phi^*_{n\eta} + B^n\Phi^*_n - 2\nu (w^{n-1})^2 \{(W^{\ n}_{\tau\eta})^2 + (W^{\ n}_{\xi\eta})^2 + \\ + (W^{\ n}_{\eta\eta})^2\} + [2\nu (w^{n-1})_\tau^{\ 2} W^{\ n}_{\eta\eta} W_\tau^{\ n} + 2\nu (w^{n-1})_\xi^{\ 2} W^{\ n}_{\eta\eta} W_\xi^{\ n} + 2\nu (w^{n-1})_\eta^{\ 2} W^{\ n}_{\eta\eta} W_\eta^{\ n}] + \\ + [-2W_\xi^{\ n} W_\eta^{\ n} + 2A_\eta^{\ n} (W_\eta^{\ n})^2 + 2A_\xi^{\ n} W_\eta^{\ n} W_\xi^{\ n} + 2A_\tau^{\ n} W_\eta^{\ n} W_\tau^{\ n} + \\ + 2W^n (B_\eta^{\ n} W_\eta^{\ n} + B_\xi^{\ n} W_\xi^{\ n} + B_\tau^{\ n} W_\tau^{\ n})] - B^n (K_1\eta + K_0) - A^n K_1 = 0\end{gathered} \tag{14}$$

Let us estimate the upper bound of the terms I_1 contained in the first set of square parentheses of (14)

$$I_1 \leqslant R_1 [(W_\tau^{\ n})^2 + (W_\xi^{\ n})^2 + (W_\tau^{\ n})^2] + \frac{\nu^2}{R_1} \{[(w^{n-1})_\tau^{\ 2}]^2 + [(w^{n-1})_\xi^{\ 2}]^2 + [(w^{n-1})_\eta^{\ 2}]^2\} (W^{\ n}_{\eta\eta})^2$$

where R_1 is some constant. The following inequality (see [5]) is valid for the function $q(x)$ which is nonnegative and which possesses bounded second derivatives for all values of x

$$(q_x)^2 \leqslant 2 \{\max |q_{xx}|\} q \tag{15}$$

Function $(w^{n-1})^2$ can be extended to embrace all the values of any of the independent variables in such a manner, that it will be nonnegative, bounded, and the modulus of its second derivative will not exceed the maximum value of the modulus of the second derivative of $(w^{n-1})^2$. Hence,

$$\frac{\nu^2}{R_1} \{[(w^{n-1})_\tau^{\ 2}]^2 + [(w^{n-1})_\xi^{\ 2}]^2 + [(w^{n-1})_\eta^{\ 2}]^2\} (W^{\ n}_{\eta\eta})^2 \leqslant \nu (w^{n-1})^2 (W^{\ n}_{\eta\eta})^2$$

provided R_1 is sufficiently large. The latter depends on the second derivatives of $(w^{n-1})^2$. Terms I_2 contained in the remaining set of square parentheses can, with help of the inequality $2ab \leqslant a^2 + b^2$, be estimated from above by means of the expression $R_2\Phi_n^* + K_7$, where R_2 depends on the first order derivatives of w^{n-1}, while K_7 is independent of n. Hence, for $\eta > \delta_2$ where $H^n = 0$, we have

$$L_n^\circ(\Phi_n) + R_3\Phi_n + K_8 \geqslant 0 \quad \text{when} \quad L_n^\circ(\Phi_n) + R^n\Phi_n \geqslant 0 \tag{16}$$

where K_8 is independent of n, while R^n depends on first and second derivatives of w^{n-1}.

To obtain the estimates of $L_n^\circ(\Phi_n)$ in Ω when $\eta \leqslant \delta_2$ we must, in addition, find

$L_n^\circ(-2W_\eta^n H^n)$. We have

$$L_n^\circ(2W_\eta^n H^n) = 2H^n L_n^\circ(W_\eta^n) + 2W_\eta^n L_n^\circ(H^n) + 4\nu(w^{n-1})^2 W_{\eta\eta}^n H_\eta^n =$$
$$= 2H^n[-\nu(w^{n-1})_\eta^2 W_{\eta\eta}^n + W_\xi^n - A_\eta^n W_\eta^n - B_\eta^n W^n - B^n W_\eta^n] +$$
$$+ 2W_\eta^n\left[L_n^\circ\left(\frac{v_0}{\nu}\right) + L_n^\circ\left(\frac{p_x}{\nu W^{n-1}}\right) - \alpha\chi(\eta)B^n W^n + \alpha W^n L^\circ(\chi) + \right.$$
$$\left. + 2\alpha\nu(w^{n-1})^2 W_\eta^n \chi'\right] + 4\nu(w^{n-1})^2 W_{\eta\eta}^n H_\eta^n \tag{17}$$

Since by Lemmas 2 and 3 we have $(w^{n-1})^2 > \gamma_0 > 0$ when $\eta \leqslant \delta_2$, terms I_1 from (14) and the term $2H^n\nu(w^{n-1})_\eta^2 W_{\eta\eta}^n$ in the expression for $L_n^\circ(-2W_\eta^n H^n)$, can be estimated with help of the inequality

$$2ab \leqslant \frac{a^2}{h} + hb^2$$

where $h > 0$ in an arbitrary number. We have

$$I_1 + 2H^n\nu(w^{n-1})_\eta^2 W_{\eta\eta}^n \leqslant {}^1/_2\nu\gamma_0(W_{\eta\eta}^n)^2 + R_4\Phi_n + K_9$$

where R_4 depends on the first derivatives of w^{n-1} and K_9 is independent of n. From (14) and (17) it follows, that, when $\eta \leqslant \delta_2$, $L_n^\circ(\Phi_n) + R_5\Phi_n + R_6 \geqslant 0$, where R_5 and R_6 are depende on w^{n-1} and its first and second derivatives.

Since $\Phi_n \geqslant 1$, $R_6\Phi_n \geqslant R_6$. Therefore $L_n^\circ(\Phi_n) + R^n\Phi_n \geqslant 0$ in Ω, Q.E.D. In order to estimate second order derivatives of w^n in Ω, we shall now consider the function

$$F_n = (W_{\tau\tau}^n)^2 + (W_{\xi\xi}^n)^2 + (W_{\tau\xi}^n)^2 + W_{\xi\eta}^n(W_{\xi\eta}^n - 2H_\xi^n) +$$
$$+ W_{\tau\eta}^n(W_{\tau\eta}^n - 2H_\tau^n) + g(\eta)(W_{\eta\eta}^n)^2 + N_0 + N_1\eta$$

where N_0 and N_1 are some constants, and a smooth function $g(\eta)$ is such, that $g(0) = 0$, $g'(0) = 0$, $g > 0$ when $\eta > 0$ and $g(\eta) = 1$ when $\eta \geqslant \delta_2$.

Lemma 5. Constants N_0 and N_1 dependent only on the first order derivatives of w^n, w^{n-1} and w^{n-2} can be chosen such, that

$$\frac{\partial F_n}{\partial\eta} \geqslant \alpha F_n - \frac{\alpha}{2}F_{n-1} \quad \text{when} \quad \eta = 0 \tag{18}$$

$$L_n^\circ(F_n) + C^n F_n + N_2 \geqslant 0 \quad \text{in } \Omega \tag{19}$$

where N_2 depends on the first order derivatives of w^n, w^{n-1} and w^{n-2} only, while C^n is dependent on w^{n-1} and its first and second order derivatives.

Proof. In the following we shall denote by C_i the constants dependent on the maxima of the moduli of w^{n-1} and of its first and second order derivatives, while N_1 will denote constants dependent only on the maxima of the moduli of first order derivatives of w^n, w^{n-1} and w^{n-2}. We shall choose $N_0 > 1$ such, that $F_n \geqslant 1$ in Ω.

Let us consider $\partial F_n/\partial\eta$ when $\eta = 0$. Using the boundary condition $W_\eta^n - H^n = 0$ when $\eta = 0$, we obtain

$$\frac{\partial F_n}{\partial\eta} = 2W_{\tau\tau}^n W_{\tau\tau\eta}^n + 2W_{\xi\xi}^n W_{\xi\xi\eta}^n + 2W_{\tau\xi}^n W_{\tau\xi\eta}^n - 2H_\tau^n H_{\tau\eta}^n - 2H_\xi^n H_{\xi\eta}^n + N_1$$

Terms $H_\tau^n H_{\tau\eta}^n$ and $H_\xi^n H_{\xi\eta}^n$ have an upper bound dependent on first order derivatives of w^n, w^{n-1} and w^{n-2}, since second order derivatives of these functions containing the differentials with respect to η can, with help of the condition $W_\eta^n - H^n = 0$, be expressed in terms of first order derivatives. Let us now estimate

$$W_{\tau\tau}^n W_{\tau\tau\eta}^n = W_{\tau\tau}^n H_{\tau\tau}^n = W_{\tau\tau}^n \left\{ \frac{v_{0\tau\tau}}{\nu} + \frac{p_{x\tau\tau}}{\nu W^{n-1}} - 2\frac{p_{x\tau} W_\tau^{n-1}}{\nu (W^{n-1})^2} + \right.$$
$$\left. + \frac{p_x}{\nu}\left[-\frac{W_{\tau\tau}^{n-1}}{(W^{n-1})^2} + 2\frac{(W_\tau^{n-1})^2}{(W^{n-1})^3} \right] + \alpha W_{\tau\tau}^n \right\} \geqslant \alpha (W_{\tau\tau}^n)^2 -$$
$$- \frac{1}{\alpha}\left[\frac{v_{0\tau\tau}}{\nu} + \frac{p_{x\tau\tau}}{\nu W^{n-1}} - 2\frac{p_{x\tau} W_\tau^{n-1}}{\nu (W^{n-1})^2} + 2\frac{p_x (W_\tau^{n-1})^2}{\nu (W^{n-1})^3} \right]^2 -$$
$$- \frac{1}{\alpha}\left[\frac{p_x}{\nu (W^{n-1})^2} \right]^2 (W_{\tau\tau}^{n-1})^2 - \frac{\alpha}{2}(W_{\tau\tau}^n)^2$$

Choice of α implies that

$$W_{\tau\tau}^n W_{\tau\tau\eta}^n \geqslant {}^1\!/_2\alpha (W_{\tau\tau}^n)^2 - {}^1\!/_4\alpha (W_{\tau\tau}^{n-1})^2 - N_3$$

Analogous estimates for $W_{\xi\xi}^n W_{\xi\xi\eta}^n$ and $W_{\xi\tau}^n W_{\xi\tau\eta}^n$, give

$$\frac{\partial F_n}{\partial \eta} \geqslant \alpha\,[(W_{\tau\tau}^n)^2 + (W_{\xi\xi}^n)^2 + (W_{\tau\xi}^n)^2] - \frac{\alpha}{2}[(W_{\tau\tau}^{n-1})^2 + (W_{\xi\xi}^{n-1})^2 + (W_{\tau\xi}^{n-1})^2] + N_1 - N_4$$

Since $W_{\eta\xi}^n (W_{\xi\eta}^n - 2H_\xi^n) + W_{\tau\eta}^n (W_{\tau\eta}^n - 2H_\tau^n)$ by virtue of the condition $W^n - H^n = 0$, $\eta = 0$ depends on first order derivatives of w^n, w^{n-1} and w^{n-2} only, we can write

$$\frac{\partial F_n}{\partial \eta} \geqslant \alpha F_n - \frac{\alpha}{2} F_{n-1} + N_1 - N_5$$

Let $N_1 = N_5$. Then, for $\eta = 0$, we obviously have

$$\frac{\partial F_n}{\partial \eta} \geqslant \alpha F_n - \frac{\alpha}{2} F_{n-1}$$

Let us now consider $L_n^\circ(F_n)$. Let F_n^* denote the sum

$$(W_{\tau\tau}^n)^2 + (W_{\xi\xi}^n)^2 + (W_{\tau\xi}^n)^2 + (W_{\xi\eta}^n)^2 + (W_{\tau\eta}^n)^2 + g\,(W_{\eta\eta}^n)^2 + N_0 + N_1\eta$$

Since $H^n = 0$ and $g(\eta) = 1$ when $\eta > \delta_2$ we have, for such η, $F_n^* = F_n$. Applying the operator

$$P \equiv 2W_{\tau\tau}^n \frac{\partial}{\partial \tau^2} + 2W_{\xi\xi}^n \frac{\partial^2}{\partial \xi^2} + 2W_{\tau\xi}^n \frac{\partial^2}{\partial \tau\, \partial \xi} + 2W_{\xi\eta}^n \frac{\partial^2}{\partial \xi\, \partial \eta} + 2W_{\tau\eta}^n \frac{\partial}{\partial \tau\, \partial \eta} + 2gW_{\eta\eta}^n \frac{\partial^2}{\partial \eta^2}$$

to both sides of the equation $L_n^\circ(W^n) + B^n W^n = 0$ we obtain

$$\nu (w^{n-1})^2 F_{n\eta\eta}^* - F_{n\tau}^* - \eta F_{n\xi}^* + A^n F_{n\eta}^* - 2\nu (w^{n-1})^2 [(W_{\tau\tau\eta}^n)^2 + (W_{\xi\xi\eta}^n)^2 + (W_{\tau\xi\eta}^n)^2 +$$
$$+ (W_{\xi\eta\eta}^n)^2 + (W_{\tau\eta\eta}^n)^2 + g(\eta)\,(W_{\eta\eta\eta}^n)^2] + \{4\nu (w^{n-1})_\tau^2 W_{\eta\eta\tau}^n W_{\tau\tau}^n +$$
$$+ 4\nu (w^{n-1})_\xi^2 W_{\eta\eta\xi}^n W_{\xi\xi}^n + 2\nu (w^{n-1})_\tau^2 W_{\eta\eta\xi}^n W_{\tau\xi}^n + 2\nu (w^{n-1})_\xi^2 W_{\eta\eta\tau}^n W_{\tau\xi}^n +$$
$$+ 2\nu (w^{n-1})_\tau^2 W_{\eta\eta\eta}^n W_{\tau\eta}^n + 2\nu (w^{n-1})_\eta^2 W_{\eta\eta\tau}^n W_{\tau\eta}^n + 2\nu (w^{n-1})_\xi^2 W_{\eta\eta\eta}^n W_{\xi\eta}^n +$$
$$+ 2\nu (w^{n-1})_\eta^2 W_{\eta\eta\xi}^n W_{\xi\eta}^n + 4g (w^{n-1})_\eta^2 W_{\eta\eta\eta}^n W_{\eta\eta}^n + 2\nu W_{\eta\eta}^n [(w^{n-1})_{\tau\tau}^2 W_{\tau\tau}^n +$$
$$+ (w^{n-1})_{\xi\xi}^2 W_{\xi\xi}^n + (w^{n-1})_{\tau\xi}^2 W_{\tau\xi}^n + (w^{n-1})_{\xi\eta}^2 W_{\xi\eta}^n + (w^{n-1})_{\tau\eta}^2 W_{\tau\eta}^n +$$
$$+ g\,(w^{n-1})_{\eta\eta}^2 W_{\eta\eta}^n] - \nu (w^{n-1})^2 g_{\eta\eta} (W_{\eta\eta}^n)^2 - 4\nu (w^{n-1})^2 g_\eta W_{\eta\eta}^n W_{\eta\eta\eta}^n - \qquad (20)$$
$$- 2W_{\tau\eta}^n W_{\tau\xi}^n - 2W_{\xi\eta}^n W_{\xi\xi}^n - 4gW_{\xi\eta}^n W_{\eta\eta}^n + P(A^n)\, W_\eta^n - g_\eta A^n (W_{\eta\eta}^n)^2 + 4A_\tau^n W_{\eta\tau}^n W_{\tau\tau}^n +$$

$$+4A_\xi{}^n W^n_{\eta\xi} W^n_{\xi\xi} + 2A_\xi{}^n W_{\eta\tau} W^n_{\tau\xi} + 2A_\tau{}^n W^n_{\eta\xi} W^n_{\tau\xi} + 2A_\xi{}^n W^n_{\xi\eta} W^n_{\eta\eta} + 2A_\eta{}^n (W^n_{\xi\eta})^2 +$$
$$+2A_\tau{}^n W^n_{\eta\tau} W^n_{\eta\eta} + 2A_\eta{}^n (W^n_{\tau\eta})^2 + 4gA_\eta{}^n (W^n_{\eta\eta})^2 + P(B^n W^n)\} - A^n N_1 = 0$$

We shall first consider the part of Ω in which $\eta \leqslant \delta_2$. By Lemmas 2 and 3, we have $(w^{n-1})^2 \geqslant \gamma_0 > 0$ when $\eta \leqslant \delta_2$. Therefore, we can use the equation $L_n{}^\circ(W^n) + B^n W^n = 0$ together with its derivative with respect to η, to express the derivatives $W^n_{\eta\eta}$ and $W^n_{\eta\eta\eta}$ for $\eta \leqslant \delta_2$ contained in the curly parentheses in (20), as a linear combination of first and second order derivatives of W^n containing not more than one differentiation with respect to η. Coefficients of these derivatives will depend on first order derivatives of w^{n-1}. After such a substitution, terms contained within the curly parentheses will consist only of the first and second order derivatives of W^n. Let us find the upper bound of these terms, using

$$2ab \leqslant a^2 + b^2 \tag{21}$$

From (20) we obtain

$$L_n{}^\circ(F_n{}^*) + C_1 F_n{}^* + C_2 + N_6 \geqslant 0$$

Here N_6 depends only on the maxima of the moduli of first order derivatives of w^n, w^{n-1} and w^{n-2}. Since $F_n{}^* \geqslant 1$ due to the choice of N_0, we have, for $\eta \leqslant \delta_2$

$$L_n{}^\circ(F_n{}^*) + C_3 F_n{}^* + N_6 \geqslant 0 \tag{22}$$

To obtain the estimate for $L_n{}^\circ(F_n)$ when $\eta \leqslant \delta_2$, we must first estimate

$$L_n{}^\circ(-2W^n_{\tau\eta} H_\tau{}^n - 2W^n_{\xi\eta} H_\xi{}^n)$$

We have

$$L_n{}^\circ(W^n_{\tau\eta} H_\tau{}^n) = L_n{}^\circ(W^n_{\tau\eta}) H_\tau{}^n + W^n_{\tau\eta} L_n{}^\circ(H_\tau{}^n) + 2\nu (w^{n-1})^2 W^n_{\tau\eta\eta} H^n_{\tau\eta} =$$
$$= H_\tau{}^n[-\nu(w^{n-1})^2_{\tau\eta} W^n_{\eta\eta} - (w^{n-1})_\tau{}^2 W^n_{\eta\eta\eta} - (w^{n-1})_\eta{}^2 W^n_{\eta\eta\tau} + W^n_{\xi\tau} -$$
$$-(B^n W^n)_{\tau\eta} - A^n_{\tau\eta} W_\eta{}^n - A_\tau{}^n W^n_{\eta\eta} - A_\eta{}^n W^n_{\eta\tau}] +$$
$$+W^n_{\tau\eta}\left[L_n{}^\circ\left(\frac{v_{0\tau}}{\nu}\right) + L_n{}^\circ\left(\left(\frac{p_x}{\nu W^{n-1}}\right)_\tau\right) + L_n{}^\circ(\alpha W_\tau{}^n \chi)\right] + 2\nu (w^{n-1})^2 W^n_{\tau\eta\eta} H^n_{\tau\eta}$$

We shall now utilise the equation $L_n{}^\circ(W^n) + B^n W^n = 0$, to replace, in the above expression, the second and third derivatives of W^n containing more than one differentiation with respect to η, with the first and second order derivatives of W^n containing not more than one differentiation with respect to η. The following expression

$$L_n{}^\circ\left(\left(\frac{p_x}{\nu W^{n-1}}\right)_\tau\right) = L_n{}^\circ\left(\frac{p_{x\tau}}{\nu W^{n-1}} - \frac{p_x W_\tau^{n-1}}{\nu (W^{n-1})^2}\right)$$

includes the first and second order derivatives of W^{n-1} and a third order derivative of the type $W^{n-1}_{\eta\eta\tau}$. The latter can be expressed in terms of first and second order derivatives of W^{n-1} and first order derivatives of w^{n-2}, using the equation obtained by differentiation of $\overset{\circ}{L}_{n-1}(W^{n-1}) + B^{n-1} W^{n-1} = 0$ with respect to τ. $L_n{}^\circ(W_{\xi\eta}{}^n H_\xi{}^n)$ is obtained in the analogous manner. Use of inequality of the type of (21), leads to

$$L_n{}^\circ(-2W^n_{\tau\eta} H_\tau{}^n - 2W^n_{\xi\eta} H_\xi{}^n) + C_4 F_n{}^* + N_7 \geqslant 0$$

for $\eta \leqslant \delta_2$. Last inequality together with (22), gives

$$L_n^\circ(F_n) + C_5 F_n^* + N_8 \geqslant 0$$

Since

$$F_n = F_n^* - 2W^n_{\tau\eta} H_\tau^n - 2W^n_{\xi\eta} H_\xi^n \geqslant 1/2\, F_n^* - N_9$$

we have, for $\eta \leqslant \delta_2$, $L_n^\circ(F_n) + C_6 F_n + N_{10} \geqslant 0$, which completes the proof.

Let us now consider $L_n^\circ(F_n)$ for $\eta \geqslant \delta_2$. For these values of η we have $F_n = F_n^*$ and $g(\eta) = 1$. Terms within the curly parentheses in (20) containing third order derivatives of W^n can be estimated using the inequality (15) in the manner used to estimate the terms I_1 in (14). Use of inequality of the type (21) on the remaining terms of the parenthesis leads to

$$L_n^\circ(F_n) + C^n F_n + N_{11} \geqslant 0 \text{ when } \eta \geqslant \delta_2$$

Theorem 1. First and second order derivatives of the solution w^n of the problem (8), (6) and (9) are uniformly bounded in n on Ω when $\tau \leqslant \tau_1$, where $\tau_1 > 0$ is a number depending on parameters of the problem (1) to (3).

Proof. We shall show that there exist numbers M_1, M_2 and $\tau_1 > 0$ such, that when $\Phi_\mu \leqslant M_1$ and $F_\mu \leqslant M_2$ when $\tau \leqslant \tau_1$ and $\mu \leqslant n-1$, then $\Phi_n \leqslant M_1$ and $F_n \leqslant M_2$ when $\tau \leqslant \tau_1$. By Lemma 4, we have $L_n^\circ(\Phi_n) + R^n \Phi_n \geqslant 0$, where R^n depends on w^{n-1} and its first and second derivatives.

Let us consider the function $\Phi_n^1 = \Phi_n e^{-\gamma\tau}$. Constant $\gamma > 0$ appearing in it will be selected later. We have $L_n^\circ(\Phi_n^1) + (R^n - \gamma)\,\Phi_n^1 \geqslant 0$ in Ω. We shall choose γ dependent on M_1 and M_2 and such, that $R^n - \gamma \leqslant -1$ in Ω^1, i.e. in Ω when $\tau < \tau_1$. Then Φ_n^1 cannot assume its greatest value within Ω^1, nor when $\xi = x_0$, $\tau = \tau_1$ or when $\eta = U(\tau, \xi)$. If Φ_n^1 assumes its greatest value when $\tau = 0$ or when $\xi = 0$, then $\Phi_n^1 = \Phi_n e^{-\gamma\tau} \leqslant \Phi_n < K_{10}$, where K_{10} is independent of n and is defined by the parameters of the problem (8), (6), and (9) only. If, on the other hand, Φ_n^1 assumes its greatest value at some point when $\eta = 0$, then at this point $\partial\Phi_n^1 / \partial\eta \leqslant 0$ and from (12) it follows that $\Phi_n^1 \leqslant 1/2\,\Phi_{n-1}^1$, i.e. $\Phi_n^1 \leqslant 1/2\,M_1$. Therefore we have

$$\Phi_n^1 \leqslant \max\{1/2\,M_1, K_{10}\} \quad \text{in } \Omega \quad \text{when } \tau \leqslant \tau_1; \qquad \Phi_n \leqslant \max\{1/2\,M_1, K_{10}\}\, e^{\gamma\tau}$$

Let τ_2 be such, that $e^{\gamma\tau_2} = 2$. If we assume that $M_1 = 2K_{10}$, then $\Phi_n \leqslant M_1$ when $\tau \leqslant \tau_2$. Let us now consider F_n. By Lemma 5, we have

$$L_n^\circ(F_n) + C^n F_n + N_2 \geqslant 0 \text{ in } \Omega$$

where C^n depends on first and second derivatives of w^{n-1} and N_2 depends on first derivatives of w^n, w^{n-1} and w^{n-2}. Let $F_n^1 = F_n e^{-\gamma_1\tau}$. Then, we have

$$L_n^\circ(F_n^1) + (C^n - \gamma_1) F_n^1 \geqslant -N_2 e^{-\gamma_1\tau} \geqslant -N_2 \text{ in } \Omega$$

Let us choose $\gamma_1 > 0$ dependent on M_1 and M_2 so, that $C^n - \gamma_1 \leqslant -1$ in Ω^2, i.e. in Ω when $\tau \leqslant \tau_2$. Then, if F_n^1 assumes its greatest value within Ω^2 either when $\tau = \tau_2$ or when $\xi = x_0$ or $\eta = U(\tau, \xi)$, then $F_n^1 \leqslant N_2(M_1)$.

If the function F_n^1 assumes its greatest value when $\tau = 0$ when $\xi = 0$, then $F_n^1 = F_n e^{-\gamma_1\tau} \leqslant F_n \leqslant N_{12}(M_1)$, where N_{12} depends on M_1. If, on the other hand, F_n^1

assumes its greatest value when $\eta = 0$, then by Lemma 5 we have at the point of maximum of F_n^1

$$0 \geqslant \frac{\partial F_n^1}{\partial \eta} \geqslant \alpha F_n^1 - \frac{\alpha}{2} F_{n-1}^1$$

and $F_n^1 \leqslant {}^1/_2 F_{n-1}^1 \leqslant {}^1/_2 F_{n-1}\, e^{-\gamma_1 \tau} \leqslant {}^1/_2 M_2$. Hence we have

$$F_n^1 \leqslant \max\{{}^1/_2\, M_2, N_{12}, N^2\} \text{ in } \Omega^2, \qquad F_n \leqslant \max\{{}^1/_2 M_2, N_{12}, N_2\}\, e^{\gamma_1 \tau}$$

Let τ_3 be such, that $e^{\gamma_1 \tau_3} = 2$. We shall take $\max\{2N_{12}, 2N_2\}$ as M_2. Then $F_n \leqslant M_2$ when $\tau \leqslant \tau_3$ and $\tau \leqslant \tau_2$. Choice of τ_3 and τ_2 depends on the constants M_1 and M_2 given previously and defined by the parameters of the problem (1) to (3).

It can be assumed that w^0 is selected so, that $\Phi_0 \leqslant M_1$ and $F_0 \leqslant M_2$. It follows that Φ_n and F_n are uniformly bounded in n when $\tau \leqslant \min\{\tau_2, \tau_3\} = \tau_1$. From the boundedness of Φ_n and F_n in n, the boundedness of first and second order derivatives of w^n follows and this proves the theorem.

Theorem 2. First and second order derivatives of the solution w^n of the problem (8), (6) and (9) are uniformly bounded in n over Ω when $\xi \leqslant \xi_1$ where ξ_1 is a number dependent only on the parameters of the problem (1) to (3) and where $\xi_1 \leqslant \xi_0$.

Proof. We shall show that there exist numbers M_1, M_2 and $\xi_1 > 0$ such, that if $\Phi_\mu \leqslant M_1$ and $F_\mu \leqslant M_2$ when $\xi \leqslant \xi_1$ and $\mu \leqslant n-1$, then $\Phi_n \leqslant M_1$ and $F_n \leqslant M_2$ when $\xi \leqslant \xi_1$.

By Lemma 4 we have $L_n^\circ(\Phi_n) + R^n \Phi_n \geqslant 0$, where R^n depends on w^{n-1} and its first and second derivatives. Let $\Phi_n = \Phi_n^1 e^{\beta\xi} \varphi_1(\beta_1 \eta)$, where $\varphi_1(s)$ is a smooth function defined by the equality $\varphi_1(s) = 2 - {}^1/_2 e^s$ for $s \leqslant \ln 3/2$ and is such, that $1 \leqslant \varphi_1 \leqslant {}^3/_2$ for all s; β and β_1 are some positive constants which shall be chosen later. We have

$$L_n^\circ(\Phi_n^1) + 2\nu\,(w^{n-1})^2 \beta_1 \frac{\varphi_1'}{\varphi_1} \Phi_{n\eta}^1 + \left(R^n - \eta\beta + A^n \beta_1 \frac{\varphi_1'}{\varphi_1} + \nu\,(w^{n-1})^2 \beta_1^2 \frac{\varphi_1''}{\varphi_1}\right)\Phi_n^1 \geqslant 0 \tag{23}$$

If $\beta_1 \eta \leqslant \ln {}^3/_2$, then $-{}^3/_4 \leqslant \varphi_1' \leqslant -{}^1/_2, \varphi_1'' \leqslant -{}^1/_2$. By Lemma 3 the inequality $(w^{n-1})^2 \geqslant \gamma_0 > 0$ is true for $\eta \leqslant \delta_2$ provided $x_0 \leqslant \xi_0$.

Let $\eta \leqslant \beta_1^{-1} \ln {}^3/_2$ and $\eta \leqslant \delta_2$. Then, constant β_1 can be selected so, that when $\xi < \xi_1$, the coefficient of Φ_n^1 in (23) satisfies the inequality

$$\left(R^n - \eta\beta + A^n \beta_1 \frac{\varphi_1'}{\varphi_1} + \nu\,(w^{n-1})^2 \beta_1^2 \frac{\varphi_1''}{\varphi_1}\right) \leqslant -1$$

In the region $\eta > \min\{\delta_2, \beta_1^{-1} \ln {}^3/_2\}$ the above inequality will be fulfilled if $\beta > 0$ is chosen sufficiently large. (Obviously, β depends on M_1 and M_2). Then, by (23), when $\xi \leqslant \xi_1$ the function Φ_n^1 cannot assume its greatest value inside Ω when $\tau = \tau_0$ or $\xi = \xi_1$ or when $\eta = U(\tau, \xi)$.

If Φ_n^1 assumes its greatest value when $\tau = 0$ or when $\xi = 0$, then

$$\Phi_n^1 = \frac{\Phi_n}{\varphi_1} e^{-\beta\xi} \leqslant \Phi_n \leqslant K_{11}$$

where K_{11} is independent of n since Φ_n can be expressed in terms of w_0, w_1 and their derivatives when $\tau = 0$ and $\xi = 0$.

If, on the other hand, Φ_n^1 assumes the greatest value when $\eta = 0$, then at this point $\partial\Phi_n^1/\partial\eta \leqslant 0$ and from (12) it follows, that

$$\Phi_n{}^1 \leqslant \frac{1}{2}\Phi^1_{n-1}, \quad \text{or} \quad \Phi_n{}^1 \leqslant \frac{1}{2}\frac{\Phi_{n-1}}{\varphi_1} e^{-\beta\xi} \leqslant \frac{1}{2} M_1$$

by virtue of previous assumption. Consequently, we have

$$\Phi_n{}^1 \leqslant \max\left\{\frac{1}{2} M_1, K_{11}\right\} \text{ in } \Omega \text{ when } \xi \leqslant \xi_1, \quad \Phi_n \leqslant \max\left\{\frac{1}{2} M_1, K_{11}\right\} \max\left[e^{\beta\xi}\varphi_1(\beta_1\eta)\right]$$

Since $\varphi_1(\beta_1\eta) \leqslant {}^3/_2$, we have $e^{\beta\xi}\varphi_1(\beta_1\eta) \leqslant 2$, if $e^{\beta\xi} \leqslant {}^4/_3$. Let us choose ξ_2 from the condition $e^{\beta\xi_2} = {}^4/_3$. Then

$$\Phi_n \leqslant \max\{M_1, 2K_{11}\} \text{ when } \xi \leqslant \xi_2$$

If we now assume that $M_1 = 2K_{11}$, then $\Phi_n \leqslant M_1$ when $\xi \leqslant \xi_2$ where ξ_2 depends on M_1 and M_2. Let us now consider F_n. By Lemma 5 we have

$$L_n{}^\circ(F_n) + C^n F_n \geqslant -N_2 \text{ in } \Omega \text{ when } \xi \leqslant \xi_0$$

Let $F_n = F_n{}^1\varphi_1(\beta_2\eta)e^{\beta_3\xi}$, where $\varphi_1(s)$ is a function defined previously. We have

$$L_n{}^\circ(F_n{}^1) + 2\nu(w^{n-1})^2\beta_2\frac{\varphi_1'}{\varphi_1}F^1_{n\eta} + \\ + \left(C^n - \eta\beta_3 + A^n\beta_2\frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2\beta_2^2\frac{\varphi_1''}{\varphi_1}\right)F_n{}^1 > -N_2\frac{e^{-\beta_3\xi}}{\varphi_1} \tag{24}$$

If $\beta_2\eta \leqslant \ln{}^3/_2$, then $-{}^3/_4 \leqslant \varphi_1' \leqslant -{}^1/_2, \varphi_1'' \leqslant -{}^1/_2$, and $1 \leqslant \varphi_1 \leqslant {}^3/_2$. By Lemma 3 we have $(w^{n-1})^2 \geqslant \gamma_0 > 0$ when $\eta \leqslant \delta_2$. Let $\eta \leqslant \min\{\delta_2, \beta_2^{-1}\ln{}^3/_2\}$. For such values of η, we can choose β_2 such, that the coefficient of F_n^1 in (24) satisfies the inequality

$$C^n - \eta\beta_3 + A^n\beta_2\frac{\varphi_1'}{\varphi_1} + \nu(w^{n-1})^2\beta_2^2\frac{\varphi_1''}{\varphi_1} \leqslant -1$$

If β_3 is sufficiently large, then this inequality will be satisfied in the region $\eta > \min\{\delta_2, \beta_2^{-1}\ln{}^3/_2\}$. Obviously, β_3 depends on M_1 and M_2. Following the reasoning adopted in the proof of Theorem 1, we obtain

$$F_n{}^1 \leqslant \max\{{}^1/_2 M_2, N_2, N_{13}\} \text{ in } \Omega \text{ when } \xi \leqslant \xi_1$$

where N_{13} depends on M_1 and where $N_{13} = \max F_n$ when $\tau = 0$ and $\xi = 0$. We have

$$F_n \leqslant \max\{{}^1/_2 M_2, N_2, N_{13}\}\max\left[e^{\beta_3\xi}\varphi_1(\beta_2\eta)\right] \leqslant \max\{M_2, 2N_2, 2N_{13}\}$$

if $e^{\beta_3\xi}\varphi_1(\beta_2\eta) \leqslant 2$ and $e^{-\beta_3\xi} \leqslant {}^4/_3$. Let us choose $M_2 = \max\{2N_2, 2N_{13}\}$ and let ξ_3 be given by $e^{\beta_3\xi_3} = {}^4/_3$. Then $F_n \leqslant M_2$ when $\xi \leqslant \xi_1$ where $\xi_1 = \min\{\xi_2, \xi_3\}$.

Boundedness of Φ_n and F_n infers the uniform boundedness in n of first and second derivatives of w^n.

Theorem 3. Functions w^n converge uniformly in Ω to the function w, which is a solution of the problem (5) to (7), provided that either $t_0 \leqslant T_1$ or $x_0 \leqslant \xi_1$.

Proof. We have shown in Theorems 1 and 2 that the first and second order derivatives of w^n in Ω are uniformly bounded in n when $t_0 \leqslant T_1$ or $x_0 \leqslant \xi_1$. We shall now prove that w^n converge in such a region of Ω, uniformly. For $v^n = w^n - w^{n-1}$, we have the equation

$$\nu(w^{n-1})^2 v^n_{\eta\eta} - v_\tau{}^n - \eta v_\xi{}^n + p_x v_\eta{}^n + \nu w^{n-1}_{\eta\eta}(w^{n-1} + w^{n-2})v^{n-1} = 0$$

with the conditions

$$v^n|_{\tau=0}=0,\quad v^n|_{\xi=0}=0,\quad v^n|_{\eta=U(\tau,\xi)}=0\quad (\nu w^{n-1}v_\eta^n - v_0 v^{n-1} + \nu w_\eta^{n-1} v^{n-1})_{\eta=0}=0$$

Let us consider a function v_1^n such, that $v^n = e^{\alpha\tau+\beta\eta}v_1^n$. We have

$$\nu(w^{n-1})^2 v_{1\eta\eta}^n - v_{1\tau}^n - \eta v_{1\xi}^n + p_x v_{1\eta}^n + \nu w_{\eta\eta}^{n-1}(w^{n-1}+w^{n-2})v_1^{n-1} + \\ + 2\nu(w^{n-1})^2\beta v_{1\eta}^n + (\nu(w^{n-1})^2\beta^2 + p_x\beta - \alpha)v_1^n = 0 \tag{25}$$

We shall choose the constant $\beta < 0$ such, that in the boundary condition for v_1 when $\eta = 0$

$$\nu w^{n-1}v_{1\eta}^n + \beta\nu w^{n-1}v_1^n + (\nu w_\eta^{n-1} - v_0)v_1^{n-1} = 0 \tag{26}$$

the coefficients of v_1^n and v_1^{n-1} satisfy the inequality

$$\max|\nu w_\eta^{n-1} - v_0| < q\nu|\beta|\min w^{n-1}(\tau,\xi,0),\qquad q<1$$

Having established β, we shall now choose $\alpha > 0$ such, that

$$\max|\nu w_{\eta\eta}^{n-1}(w^{n-1}+w^{n-2})| < q(\alpha - \max|\nu(w^{n-1})^2\beta^2 + p_x\beta|)$$

Then, if $|v^n|$ attains its greatest value at some internal or boundary point of Ω, from (25) and (26) it follows that $\max|v_1^n| \leqslant q\max|v_1^{n-1}|$, i.e. sum of the series $v_1^1 + v_1^2 + \ldots + v_1^n + \ldots$, partial sums of which are equal to $w^n e^{-\alpha\tau-\beta\eta}$, is smaller than the sum of the geometrical progression, and is, therefore, uniformly convergent. The boundedness of w^n and its first and second derivatives implies uniform convergence of all first derivatives of w^n as $n\to\infty$. From (8) it follows that w^n also converge uniformly as $n\to\infty$, provided that $\eta < U(\tau,\xi) - \delta_3$, where $\delta_3 > 0$ is arbitrary.

Thus we have shown that solution of the problem (5) to (7) exists in Ω if x_0 or t_0 are sufficiently small and, provided that solution of the problem (8), (6) and (9) exists.

We shall now show one of the methods of constructing w^n. (We should note that analogous methods were utilised in investigation of linear equations of the type (8) in [5]). Below we shall give a boundary problem for an elliptic equation in a special region, the solutions $w^{\varepsilon n}$ of which converge uniformly to w^n as $\varepsilon \to 0$. A corresponding boundary problem for a parabolic equation can be constructed in the analogous manner.

Let G be an infinitely differentiable bounded region in the $\xi\eta$-plane such, that a cylinder $[0, t_0]\times G$ contains Ω and the boundary σ of G contains a segment $[-2\delta, x_0+2\delta]$ of the ξ-axis, where $\delta > 0$ is a small number.

We shall assume that in some vicinity of the point A of intersection of σ with the straight line $\xi = 0$, σ lies on the straight line $\eta = \eta_1 = \text{const}$. Let us consider a singly connected infinitely differentiable region Q whose boundary S coincides with the cylinder $[-1, t_0+1]\times G$ when $-1 \leqslant \tau \leqslant t_0+1$, Q being interior to the cylinder $[-2, t_0+2]\times G$. We shall denote by Ω_1 these points of Q, for which either $\tau \geqslant 0$ and $\xi \geqslant 0$, or $\tau \geqslant t_0$. Let us also extend smoothly the coefficient p_x from (8) and the functions v_0 and p_x from (9), to all values of ξ and τ. We shall denote by S_1 the boundary $\{\tau = 0,\ 0\leqslant\xi\leqslant x_0,\ 0\leqslant\eta\leqslant U(0,\xi)\}$ of the region Ω, $S_2 = \{0\leqslant\tau\leqslant t_0,\ \xi=0,\ 0\leqslant\eta\leqslant U(\tau,0)\}$ and $S_0 = \{0\leqslant\tau\leqslant t_0,\ 0\leqslant\xi\leqslant x_0,\ \eta=0\}$.

We shall also assume that a smooth function w^* exists, defined in $Q - \Omega_1$ and satisfying the conditions

$$w^*|_{\tau=0} = w_0 \quad \text{on } S_1, \quad w^*|_{\xi=0} = w_1 \quad \text{on } S_2$$

$L(w^*) = 0(\xi^4)$ near S_2 when $\xi \leqslant 0$ and $\tau \geqslant 0$

$L(w^*) = O(\tau^4)$ near S_1 when $\xi \geqslant 0$ and $\tau \leqslant 0$

$l(w^*) = O(\xi^4)$ on S near the segment $[0, t_0]$ of the τ-axis

$l(w^*) = O(\tau^4)$ on S near the segment $[0, x_0]$ of the ξ-axis

It can be assumed that w^* has continuous sixth order derivatives in the closed region $\overline{Q - \Omega_1}$ and is an infinitely differentiable function outside some neighborhood of the boundaries S_1 and S_2 of the region Ω. Such a function w^* can be constructed if w_0, w_1, v_0 and p_x are sufficiently smooth and if, apart from that, w_0 and w_1 satisfy the conditions,on the τ, ξ- and η-axes, of the problem (5) to (7).

For example, w^* can be constructed as follows. We shall assume, that in the vicinity of S_2 when $\xi \leqslant 0$ and $\tau \geqslant 0$,

$$w^* = w_1 + \xi \frac{\partial w}{\partial \xi}\Big|_{\xi=0} + \cdots + \frac{\xi^m}{m!}\frac{\partial^m w}{\partial \xi^m}\Big|_{\xi=0}, \qquad m \geqslant 4 \tag{27}$$

Here derivatives of w with respect to ξ when $\xi = 0$, can be found from (5) and from the equations obtained from it by differentiation with respect to ξ under the condition that $w = w_1$ when $\xi = 0$. When $\tau \leqslant 0$ and $\xi \geqslant 0$ near the boundary S_1 of Ω, function w^* can be found from

$$w^* = w_0 + \tau \frac{\partial w}{\partial \tau}\Big|_{\tau=0} + \cdots + \frac{\tau^m}{m!}\frac{\partial^m w}{\partial \tau^m}\Big|_{\tau=0}, \qquad m \geqslant 4 \tag{28}$$

where derivatives of w with respect to τ when $\tau = 0$ can be found from (5) and from the equations obtained from it by differentiation with respect to τ, provided that $w = w_0$ when $\tau = 0$. It is easy to see that the function w^* given by (27) and (28) near the boundary of Ω lying on the planes $\tau = 0$ and $\xi = 0$ and extended in an arbitrary smooth manner into the remaining part of the region $Q - \Omega_1$, satisfies the imposed conditions provided that w_0 and w_1 are sufficiently smooth and fulfill the conditions of compatibility on the τ-, ξ- and η-axes. When constructing the functions w^n satisfying Equation (8) and conditions (6) and (9), we shall use w^* extended in an arbitrary smooth manner to Ω_1, as w^0. We shall assume that the function w^{n-1} possessing bounded derivatives of the fourth order in Q which is a solution of (8), (6) and (9) in Ω is already constructed and we shall try to determine w^n. It will be shown that $w^n = w^*$ in $Q - \Omega_1$ if $w^{n-1} = w^*$ in $Q - \Omega_1$. Let $\sigma_\delta = \sigma - q_\delta$ where q_δ is a segment $[-2\delta, x_0 + 2\delta]$ of the ξ-axis and let $S^\delta = [-1, t_0 + 1]\sigma_\delta$. We shall consider the operator

$$L^\varepsilon(w) \equiv \varepsilon(w_{\tau\tau} + w_{\xi\xi} + w_{\eta\eta}) + a_1 w_{\tau\tau} + a_2 w_{\xi\xi} + a_3 w_{\eta\eta} + \nu (w^{n-1})_\varepsilon^2 w_{\eta\eta} - \\ - w_\tau - \eta w_\xi + (p_x)_\varepsilon w_\eta - 2(a_1 + \varepsilon) w$$

in Q. Here $\varepsilon > 0$, the infinitely differentiable functions a_1, a_2 and a_3 are positive when $\tau < -\frac{1}{2}$ and when $\tau > t_0 + \delta$, a_3 is also positive in the δ-neighborhood of S, while a_2 is positive everywhere in this neighborhood except at the points lying on the plane $\xi = 0$ when $0 \leqslant \tau \leqslant t_0$. At the remaining points of Q, functions a_1, a_2 and a_3 are equal to zero. We choose δ small enough to ensure that a_1, a_2 and a_3 are equal to zero in Ω $(\Psi)_\varepsilon$ will denote the mean value of ψ within a circle of radius ε , where a positive, infinitely differentiable kernel is used in the averaging process.

Consider, in Q, a boundary problem for the elliptic equation

$$L^\varepsilon(w) = (f)_\varepsilon \tag{29}$$

with the following boundary condition on S

$$\frac{\partial w}{\partial n} = (F)_\varepsilon \tag{30}$$

where n is a vector normal to S. Function f appearing in (29), is defined in Q thus:

$$f = L(w^*) + a_1 w_{\tau\tau}^* + a_2 w_{\xi\xi}^* + a_3 w_{\eta\eta}^* - 2a_1 w^*$$

in $Q - \Omega_1$, $f = 0$ in Ω and is an arbitrary smooth continuation of this function (with bounded fourth order derivatives) in the remainder of Q. Function F is

$$\frac{v_0}{\nu} + \frac{p_x}{\nu w^{n-1}} \quad \text{on } S_0, \qquad F = \frac{\partial w^*}{\partial n} \quad \text{on } \gamma$$

Here γ is the intersection of S with the boundary of $Q - \Omega_1$. On the remainder of S, function F appearing in (30) will be an arbitrary smooth continuation of F given on S_0 and γ.

Obviously it can be assumed by virtue of the properties of w^*, that function f has, in Q, bounded derivatives of up to and including the fourth order and is infinitely differentiable outside the δ-neighborhood of Ω, while F has bounded fourth order derivatives in some neighborhood of S_0 and is infinitely differentiable on the remainder of S. The boundary problem (29) and (30) has a unique solution $w^{\varepsilon n}$ in Q, and since the boundary of Q, coefficients of the equation (29) and the right-hand sides in (29) and (30) are infinitely differentiable, it follows that $w^{\varepsilon n}$ is an infinitely differentiable function in the closure of Q (see e.g. [6]). Uniqueness of the solution to the problem (29) and (30) follows from the maximum principle [7]. We shall now show that $w^{\varepsilon n}$ and their derivatives up to and including the fourth order, are uniformly bounded in ε.

Lemma 6. Solution $w^{\varepsilon n}$ of the problem (29) and (30) in the region Q, are uniformly bounded in ε.

Proof. Let us make a substitution

$$w^{\varepsilon n} = v^\varepsilon \psi^1$$

in (29), where $\psi^1(\tau) = 1$ when $\tau \leqslant -1$ and $\psi^1(\tau) = 1 + b(1+\tau)^3$ when $-1 \leqslant \tau \leqslant t_0 + 2$. Constant $b > 0$ shall be chosen so, that $\psi_{\tau\tau}^1 \leqslant \psi^1$ in Q. Let $6b(t_0 + 3) < 1$. For the function v^ε, we shall have in Q

$$\begin{gathered}\varepsilon\Delta v^\varepsilon + a_1 v_{\tau\tau}^\varepsilon + a_2 v_{\xi\xi}^\varepsilon + a_3 v_{\eta\eta}^\varepsilon + \nu (w^{n-1})_\varepsilon^2 v_{\eta\eta}^\varepsilon - v_\tau^\varepsilon - \eta v_\xi^\varepsilon + (p_x)_\varepsilon v_\eta^\varepsilon + \\ + 2(a_1 + \varepsilon)\frac{\psi_\tau^1}{\psi^1} v_\tau^\varepsilon + \left[(a_1 + \varepsilon)\frac{\psi_{\tau\tau}^1}{\psi^1} - \frac{\psi_\tau^1}{\psi^1} - 2(a_1 + \varepsilon)\right] v^\varepsilon = \frac{(f)_\varepsilon}{\psi^1}\end{gathered} \tag{31}$$

and the boundary condition on S

$$\frac{\partial v^\varepsilon}{\partial n} = \frac{(F)_\varepsilon}{\psi^1} \quad \text{when} \quad -2 \leqslant \tau \leqslant t_0 + 1 \tag{32}$$

$$\frac{\partial v^\varepsilon}{\partial n} + \frac{\partial \psi^1 / \partial n}{\psi^1} v^\varepsilon = \frac{(F)_\varepsilon}{\psi^1} \quad \text{when} \quad \tau \geqslant t_0 + 1 \tag{33}$$

Since

$$\frac{\partial\psi^1}{\partial n} = \psi_\tau^1 \frac{\partial\tau}{\partial n} \leqslant 0 \quad \text{when } \tau \geqslant t_0 + 1 \quad \text{on} \quad S$$

the coefficient of v^ε in (33) is nonpositive. (Q can be assumed convex when $\tau \geqslant t_0 + 1$). Coefficient of v^ε in (31) is negative. Indeed, $-(a_1+\varepsilon) + (a_1+\varepsilon)\psi_{\tau\tau}^1/\psi^1 \leqslant 0$, since $\psi_{\tau\tau}^1/\psi^1 \leqslant 1$, and $\psi_\tau^1 > 0$ when $\tau > -1$ and $a_1 > 0$ when $\tau < -\frac{1}{2}$. Applying the estimate proved in Theorem 4 of [7] to the solution of the elliptic equation (31) with the boundary condition (32) and (33) we shall find, that v^ε, and consequently $w^{\varepsilon n}$, are uniformly bounded in ε over Q.

Lemma 7. Solutions $w^{\varepsilon n}$ of the problem (29) and (30) possess, in Q, derivatives up to and including the fourth order, uniformly bounded in ε.

Proof. We first note that in Q, when $\tau > t_0 + \delta + r_1$ and when $\tau < -1/2 - r_1$, where r_1 is an arbitrary positive number, equation (29) is uniformly elliptic with respect to ϵ. Consequently, in agreement with well known *a*-priori Schauder type estimates (see e.g. [6]), the derivatives of $w^{\varepsilon n}$ of order m are uniformly bounded in ε with respect to their moduli when $\tau > t_0 + \delta + r_1$ and when $\tau < -1/2 - r_1$, provided that w^{n-1} possess bounded derivatives of the $(m-1)$th order in that region.

Let the point $P(\xi, \eta)$ belong to σ_δ where $|\xi| \geqslant 2\delta$ and let A_δ denote its δ-neighborhood on the $\xi\eta$-plane. We shall consider the cylinder

$$B_\delta = [-1/2 - r_1, t_0 + \delta + r_1] \times A_\delta.$$

and we shall show, that in this region, $w^{\varepsilon n}$ possess derivatives of up to the fourth order inclusive, uniformly bounded in ε. It can be assumed that in B_δ, the coefficient a_1 depends only on τ, while a_2 and a_3 depend only on ξ and η. We shall pass to new coordinates ξ' and η' in A_δ in such a manner, that the boundary belonging to A_δ will transform into a straight line $\eta' = 0$, while the direction n of the normal to σ will become the direction of the η'-axis. Boundary condition (29) will, in new coordinates which we shall from now on denote by ξ and η, assume the form $\partial w^{\varepsilon n}/\partial\eta = F_\varepsilon^*$.

Let $T(\tau, \xi, \eta)$ be a function in B_δ such, that $\partial T/\partial\eta = F_\varepsilon^*$ when $\eta = 0$. Function $z = w^{\varepsilon n} - T$ satisfies in B_δ, the equation

$$M(z) \equiv (\varepsilon + a_1)\, z_{\tau\tau} - z_\tau + a_{11} z_{\xi\xi} + 2a_{12} z_{\xi\eta} + a_{22} z_{\eta\eta} + b_1 z_\xi + b_2 z_\eta - 2(\varepsilon + a_1)\, z = f_\varepsilon^* \tag{34}$$

and the condition $z_\eta = 0$ on S. At the same time $a_{11}\alpha_1^2 + 2a_{12}\alpha_1\alpha_2 + a_{22}\alpha_2^2 \geqslant \lambda_0(\alpha_1^2 + \alpha_2^2)$.

In order to obtain an estimate of first order derivatives of z with respect to ξ and η, we shall consider the function

$$\Lambda^1 = \rho_\delta^2(\xi,\eta)\,[z_\xi^2 + z_\eta^2] + c_1 z^2 + c_2\eta, \qquad c_2 > 0$$

Here constant c_1 is assumed to be sufficiently large and will be selected later, while $\rho_\delta(\xi,\eta)$ is a function equal to unity in $A_{\delta/2}$ and equal to zero in some small region near the boundary of A_δ not belonging to σ. Also, $\rho_{\delta\eta} = 0$ on σ.

It is easily seen that $\partial\Lambda^1/\partial\eta = c_2 > 0$ on S, consequently Λ^1 cannot assume its greatest value on S. If Λ^1 attains its maximum at the points on the boundary of B_δ where $\rho_\delta = 0$, then

$$\Lambda^1 \leqslant \max\,[c_1 z^2 + c_2\eta] \leqslant c_3$$

where c_3 is independent of ε. It can easily be checked that for sufficiently large value of c_1, $M(\Lambda^1) - \Lambda^1 \geqslant -c_4$ in B_δ, provided c_4 is sufficiently large. Hence, if Λ^1 assumes its greatest value inside B_δ, then $\Lambda^1 < c_4$. When $\tau = t_0 + \delta + r_1$ and $\tau = -1/_2 - r_1$ then Λ^1 is uniformly bounded in ϵ, the fact which we have already established. Since Λ^1 is uniformly bounded in ε in B_δ, therefore z_ξ and z_η are bounded in

We shall represent (34) as follows (B_{δ_1}, $\delta_1 < \delta$).

$$M(z) \equiv \Gamma(z) + M^1(z) = f_\varepsilon^*, \qquad \Gamma(z) \equiv (\varepsilon + a_1) z_{\tau\tau} - z_\tau$$

It can be assumed that the coefficients of the operator M^1 are independent of τ. Consequently, $\Gamma(z)$ satisfies the equation

$$M(\Gamma) \equiv \Gamma(\Gamma) + M^1(\Gamma) = \Gamma(f_\varepsilon^*) \text{ in } B_\delta, \ \Gamma_\eta|_{\eta=0} = 0 \quad \text{on } S \tag{35}$$

Consider, in B_{δ_1}, a function

$$\Lambda^2 = \rho_{\delta_1}^2 [z_{\xi\xi}^2 + z_{\xi\eta}^2 + \Gamma^2(z)] + c_5(z_\xi^2 + z_\eta^2) + c_6\eta$$

Using (34) and (35) we easily obtain

$$M(\Lambda^2) - \Lambda^2 \geqslant -c_7 \quad \text{in } B_{\delta_1}, \quad \frac{\partial \Lambda^2}{\partial \eta} = c_6 > 0$$

on S, provided $c_5 > 0$ is sufficiently large. From this it follows, that Λ^2 is uniformly bounded in ε over B_{δ_1}, while $\Gamma(z)$, $z_{\xi\xi}$ and $z_{\xi\eta}$ are uniformly bounded in ε over B_{δ_2}, $\delta_2 < \delta_1$. From (34) it follows that $z_{\eta\eta}$ is also uniformly bounded in ϵ. Considering the equation for z_τ of the form $(a_1 + \varepsilon) z_{\tau\tau} - z_\tau = \Gamma$ and taking into account the boundedness of Γ in B_{δ_2} and of z_τ when $\tau = -1/_2 - r_1$ and $\tau = t_0 + \delta + r_1$, we reach the conclusion that z_τ is also uniformly bounded with respect to ε, in B_{δ_2}.

Since the function $\Gamma(z)$ is bounded in B_{δ_2} and satisfies (35) with the boundary condition $\Gamma_\eta|_{\eta=0} = 0$ we can, for Γ and B_{δ_2}, consider the functions Λ^1 and Λ^2 just as it was done for z, and obtain the estimates uniform with respect to ε in B_{δ_3} ($\delta_3 < \delta_2$), for the following derivatives

$$\Gamma_\xi, \quad \Gamma_\eta, \quad \Gamma_{\xi\xi}, \quad \Gamma_{\xi\eta}, \quad \Gamma(\Gamma), \quad \Gamma_{\eta\eta}, \quad \Gamma_\tau$$

Differentiating (35) with respect to τ we obtain, for Γ_τ,

$$(a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a_1')\Gamma_{\tau\tau} + M^1(\Gamma_\tau) = (\Gamma(f_\varepsilon^*))_\tau$$

together with the condition $\Gamma_{\tau\eta}|_{\eta=0} = 0$ on S. By definition, $a_1'(\tau)$ is small in B_δ. Therefore, equation for Γ_τ has the same form as (35). Hence, the derivatives of Γ of the type

$$\Gamma_{\tau\xi}, \quad \Gamma_{\tau\eta}, \quad \Gamma_{\tau\xi\xi}, \quad \Gamma_{\tau\eta\xi}, \quad (a_1 + \varepsilon)\Gamma_{\tau\tau\tau} - (1 - a_1')\Gamma_{\tau\tau}, \quad \Gamma_{\tau\eta\eta}, \quad \Gamma_{\tau\tau}$$

can be estimated uniformly with respect to ε in B_{δ_4}, ($\delta_4 < \delta_3$), in the manner adopted previously for z. Analogous considerations for $\Gamma_{\tau\tau}$, yield, in B_{δ_5}, ($\delta_5 < \delta_4$), uniform in ε boundedness of derivatives

$$\Gamma_{\tau\tau\xi}, \quad \Gamma_{\tau\tau\eta}, \quad \Gamma_{\tau\tau\xi\xi}, \quad \Gamma_{\tau\tau\eta\xi}, \quad (a_1 + \varepsilon)\Gamma_{\tau\tau\tau\tau} - (1 - 2a_1')\Gamma_{\tau\tau\tau}, \quad \Gamma_{\tau\tau\eta\eta}, \quad \Gamma_{\tau\tau\tau}$$

from which it follows, that in B_{δ_5}, third and fourth order derivatives of z containing more than one differentiation with respect to τ and uniformly bounded in ε together with first derivatives of $\Gamma(\Gamma)$ with respect to ξ and η, satisfy the Lifschitz condition with respect to ξ and η, uniformly in ε and τ. From the Schauder type estimates (see [6]) for the elliptic equation,

$$M^1(\Gamma) = -\Gamma(\Gamma) + \Gamma(f_\varepsilon^*)$$

it follows, that the derivatives of up to and including third order of Γ with respect to ξ and η are bounded, and satisfy the Hölder condition uniformly with respect to ε and τ in B_{δ_6} ($\delta_6 < \delta_5$). Schauder type estimates for (34) for z written in the form

$$M^1(z) = -\Gamma(z) + f_\varepsilon^*$$

lead to the conclusion, that z possesses derivatives with respect to ξ and η of up to and including the fourth order uniformly bounded in ε and τ on B_{δ_7}, $\delta_7 < \delta_6$. In this manner we have obtained the estimates of derivatives of $w^{\varepsilon n}$ with respect to τ, ξ and η of up to and including the fourth order in some neighborhood of the whole of S with exception of the neighborhood of S_0 and of the neighborhood ω of the intersection of S with the plane $\xi = 0$, lying in the plane $\eta = \eta_1$.

We shall now introduce, in (29) and (30), a new function W, defined by

$$w = We^{\varphi_2(\eta)}, \qquad \varphi_2 = -\alpha\eta(\eta_1 - \eta)/\eta_1, \ \alpha = \text{const} > 0$$

For W, we shall have the following boundary conditions

$$\frac{\partial W}{\partial \eta} - \alpha W = (F)_\varepsilon \text{ when } \eta = 0, \qquad -\frac{\partial W}{\partial \eta} - \alpha W = (F)_\varepsilon \text{ when } \eta = \eta_1$$

In order to estimate in Q first order derivatives of $w^{\varepsilon n}$, we shall consider, in Q, when $-{}^1/_2 - r_1 \leqslant \tau \leqslant t_0 + \delta + r_1$ (calling this region Q_{r1}), a function

$$X_1 = W_\xi^2 + W_\tau^2 + W_\eta(W_\eta - 2Y) + k(\eta), \qquad Y = (\alpha W + (F)_\varepsilon)\varkappa_1(\eta)$$
$$\varkappa_1(\eta) = 1 \quad \text{when } |\eta| < \delta$$
$$\varkappa_1(\eta) = -1 \quad \text{when } |\eta - \eta_1| < \delta$$
$$\varkappa_1(\eta) = 0 \quad \text{when } 2\delta < \eta < \eta_1 - 2\delta$$

Here $k(\eta)$ is a positive function, which shall be specified later. Obviously, on the boundary S lying in the plane $\eta = 0$ or $\eta = \eta_1$, the equality $\partial W/\partial\eta - Y = 0$. holds. We have

$$\left.\frac{\partial X_1}{\partial \eta}\right|_{\eta=0} = 2W_\xi W_{\xi\eta} + 2W_\tau W_{\tau\eta} - 2W_\eta Y_\eta + k'(0) =$$
$$= 2\alpha[W_\xi^2 + W_\tau^2] - 2YY_\eta + 2W_\xi(F)_{\varepsilon\xi} + 2W_\tau(F)_{\varepsilon\tau} + k'(0) > 0$$

provided $k'(0) > 0$ is sufficiently large. Analogously, having selected in X_1 a function $k(\eta)$ so, that $k'(\eta_1) < 0$ and has a sufficiently large modulus, we find that $\partial X_1/\partial\eta|_{\eta=\eta_1} < 0$. Approach employed in the proof of Lemma 4, yields

$$L^{\circ\varepsilon}(X_1) + c_8 X_1 \geqslant -c_9$$

$$L^{\circ\varepsilon}(W) \equiv L^\varepsilon(W) + 2[(\varepsilon + a_8) + \nu(w^{n-1})_\varepsilon^2]\varphi_{2\eta}\frac{\partial W}{\partial\eta} + \tag{36}$$
$$+ \{(\nu(w^{n-1})_\varepsilon^2 + \varepsilon + a_8)[\varphi_{2\eta\eta} + (\varphi_{2\eta})^2] + (p_x)_\varepsilon\varphi_{2\eta}\}W$$

Here c_8 and c_9 are independent of ε. Let us consider in Q_{r1}

$$X_1^* = X_1 e^{-\beta t}, \qquad \beta = \text{const} > 0$$

If β is sufficiently large, then the coefficient of X_1^* in (36) is negative and smaller than -1. From (36) it follows that if X_1^* assumes its greatest value within Q_{r1}, then X_1^* has an upper bound independent of ε. X_1^* cannot assume its greatest value when $\eta = 0$ and $\eta = \eta_1$; on the remainder of the boundary of Q_{r1} function X_1^* is uniformly bounded in ε by virtue of the previous estimates. Estimation uniform in ε, of the second and third order derivatives of $w^{\varepsilon n}$ proceeds analogously by considering the functions

$$X_2 = W_{\tau\tau}^2 + W_{\xi\xi}^2 + W_{\tau\xi}^2 + W_{\eta\xi}(W_{\eta\xi} - 2Y_\xi) + W_{\eta\tau}(W_{\eta\tau} - 2Y_\tau) + g_1^2(\eta) W_{\eta\eta}^2 + k(\eta)$$

$$X_3 = (X_3)' + g_1^2(\eta)[W_{\eta\eta\eta}^2 + W_{\eta\eta\xi}^2 + W_{\eta\eta\tau}^2] + W_{\eta\xi\xi}(W_{\eta\xi\xi} - 2Y_{\xi\xi}) + W_{\eta\tau\tau}(W_{\eta\tau\tau} - 2Y_{\tau\tau}) + W_{\eta\xi\tau}(W_{\eta\xi\tau} - 2Y_{\xi\tau}) + k(\eta)$$

$g_1(\eta) = 0$ when $\eta < \delta/2$, $g_1(\eta) = 0$ when $\eta > \eta_1 - \delta/2$, $g_1(\eta) = 1$ when $\eta_1 - \delta > \eta > \delta$

Here $(X_3)'$ is sum of the squares of third derivatives of W with respect to ξ and τ. Estimates of X_2 and X_3 can be obtained in the manner similar to that used for X_1, but in derivation of the inequality of the type of (36) for X_2 and X_3, use should be made of the fact, that the coefficient of $W_{\eta\eta}$ in (29) is positive when $\eta < \delta$ and $\eta_1 - \eta < \delta$, just as in the proof of Lemma 5.

When estimating the fourth order derivatives of W, we should turn our attention to the following. Let us consider the function

$$X_4 = (X_4)' + g_1^2(\eta)(X_4)'' + W_{\eta\xi\xi\xi}(W_{\eta\xi\xi\xi} - 2Y_{\xi\xi\xi}) + W_{\eta\tau\tau\tau}(W_{\eta\tau\tau\tau} - 2Y_{\tau\tau\tau}) + W_{\eta\xi\xi\tau}(W_{\eta\xi\xi\tau} - 2Y_{\xi\xi\tau}) + W_{\eta\tau\tau\xi}(W_{\eta\tau\tau\xi} - 2Y_{\tau\tau\xi}) + k(\eta)$$

where $(X_4)'$ is the sum of the squares of fourth order derivatives of W not differentiated with respect to η and $(X_4)''$ is the sum of squares of the fourth order derivatives differentiated more than once with respect to η.

Function X_4 includes third order derivatives of Y, hence also of $(F)_\varepsilon$. Operator $L^{\circ\varepsilon}(X_4)$ can be estimated in terms of $L^{\circ\varepsilon}(Y_{\tau\tau\tau})$, $L^{\circ\varepsilon}(Y_{\xi\xi\xi})$, $L^{\circ\varepsilon}(Y_{\tau\tau\xi})$ and $L^{\circ\varepsilon}(Y_{\tau\xi\xi})$, containing fifth order derivatives of $(F)_\varepsilon$. By virtue of its construction, function F is infinitely differentiable outside the δ-neighborhood of S_0 and possesses fourth order bounded derivatives on S. In the region Q belonging to the δ-neighborhood of S_0, operator $L^{\circ\varepsilon}$ contains second order differentials in ξ and τ with the coefficient ε of the type $\varepsilon(\partial^2/\partial\tau^2)$ and $\varepsilon(\partial^2/\partial\xi^2)$. Since F has fourth order bounded derivatives, therefore fifth order derivatives of the averaged function $(F)_\varepsilon$ are of the order of $1/\varepsilon$. Consequently, application of the operator $L^{\circ\varepsilon}$ to third order derivatives of $(F)_\varepsilon$ gives, as a result, a quantity bounded in ε. The remainder of the procedure of obtaining the estimate for X_4 follows that employed for X_1, X_2 and X_3. Thus we obtain the final result, that the derivatives of $w^{\varepsilon n}$ of up to and including the fourth order, are uniformly bounded in ε.

Theorem 4. When $\varepsilon \to 0$, solutions $w^{\varepsilon n}$ of the problem (29) and (30) in the region Q, converge to the solution of w^n of the problem (8), (6) and (9) in Ω.

Proof. By Lemma 7, the derivatives of $w^{\varepsilon n}$ of up to and including the fourth order are uniformly bounded in ε. Consequently, a sequence $w^{\varepsilon_k n}$ can be chosen such, that as as $\varepsilon_K \to 0$, functions $w^{\varepsilon n}$ converge uniformly to w^n in Q, together with their derivatives of up to and including the third order. Obviously, the limit function w^n satisfies, in Q, Equation (8) and the boundary condition (9), when $\eta = 0$. We shall show now, that w^n satisfies the conditions (6). To do this, we shall have to prove that $w^n = w^*$ in $Q - \Omega_1$.

Let $w^n - w^* = Z$. By definition, we have in $Q - \Omega_1$

$$a_1 Z_{\tau\tau} + a_2 Z_{\xi\xi} + a_3 Z_{\eta\eta} + \nu (w^*)^2 Z_{\eta\eta} - Z_\tau - \eta Z_\xi + p_x Z_\eta - 2a_1 Z = 0$$

and $\partial z / \partial n = 0$ on the part of the boundary of $Q - \Omega_1$, which belongs to S. Let us consider, in $Q - \Omega_1$, function Z^* defined by $Z = Z^* \psi^1(\tau)$ where ψ^1 is a function constructed in the proof of Lemma 6. We shall obtain for Z^* an equation in $Q - \Omega_1$, in which the coefficient of Z^* will be strictly negative in the closure of $Q - \Omega_1$. Let $E(\tau, \xi, \eta)$ be a smooth function in Q such, that $\partial E / \partial n < 0$ on S and $E > 1$. Consider the function $Z^1 = Z^*(E + c)$ where c is a positive constant. In the equation obtained for Z^1 the coefficient of z^1 will be negative, provided c is sufficiently large. Boundary condition on S is $\partial Z^1 / \partial n - \alpha_1 Z^1 = 0$, where $\alpha_1 = -\partial E / \partial n > 0$. Modulus of Z^1 cannot assume its greatest value on S, since at the maximum of $|Z^1|$ on S we have $Z^1 (\partial Z^1 / \partial n) - \alpha_1 (Z^1)^2 < 0$, which contradicts the boundary condition on S. Maximum of $|Z^1|$ cannot also be achieved inside $Q - \Omega_1$, since at the maximum of $|Z^1|$ we have $Z_\tau^1 = 0$, $Z_\xi^1 = 0$, $Z_\eta^1 = 0$, $Z^1 Z_{\eta\eta}^1 \leqslant 0$, $Z^1 Z_{\xi\xi}^1 \leqslant 0$, $Z^1 Z_{\tau\tau}^1 \leqslant 0$, which contradicts the fact that at this point the equation obtained for Z^1 is satisfied.

It can be shown in the analogous manner that the maximum of $|Z^1|$ cannot be reached on the boundary of $Q - \Omega_1$ when $\tau = 0$ or $\xi = 0$. Consequently $Z^1 \equiv 0$ in $Q - \Omega_1$, from which it follows that $w^n = w^*$ in $Q - \Omega_1$. Hence $w^n|_{\tau=0} = w_0$ and $w^n|_{\xi=0} = w_1$.

We shall now show that $w^n = 0$ on the surface $\eta = U(\tau, \xi)$. From previous arguments it follows, that $w^n = 0$ when $\tau = 0$ and $\eta = U(0, \xi)$, and also $w^n = 0$ when $\xi = 0$ and $\eta = U(\tau, 0)$. Since $w^{n-1} = 0$ on the surface $\eta = U(\tau, \xi)$, w^n satisfies, on this surface, the equation $w_\tau^n + \eta w_\xi^n - p_x w_\eta^n = 0$. We have said before that the direction $(1, \eta, -p_x)$ lies on the plane tangent to the surface $\eta = U(\tau, \xi)$. These directions form a vector field on this surface. Integral curves of this field intersect, on continuation to smaller values of τ, the boundary of the surface either when $\xi = 0$ or when $\tau = 0$, and we have there $w^n = 0$. Since w^n is constant on these integral curves, $w^n = 0$ on the whole of the surface $\eta = U(\tau, \xi)$. We should note, that the constructed function w^n possesses, in Ω, third order derivatives satisfying the Lifschitz condition.

Let us now return to the initial problem (1) to (3). We consider fulfilled all the previous assumptions of sufficient smoothness of p, v_0, u_1, u_0, w_0, and w_1 and conditions of compatibility of these functions, from which the existence of the function w^* shown above, can be inferred.

Theorem 5. There exists a unique solution of the problem (1) to (3) in the region D, provided that either $t_0 \leqslant \tau_1$, or $x_0 \leqslant \xi_1$ where $\tau_1 > 0$ and $\xi_1 > 0$ are some numbers defined by the data of the problem (1) to (3). This solution has the following properties: $u > 0$ when $y > 0$, $u_y > 0$ when $y \geqslant 0$, derivatives u_t, u_x, u_y, and u_{yy} are continuous and bounded in D. Also, u_{yy} / u_y and $(u_{yyy} u_y - u_{yy}^2) / u_y^3$ are bounded in D.

Proof. Let w be solution of the problem (5) to (7) constructed in the course of proof of Theorem 4. We shall determine u using the condition $w = u_y$, or

$$y = \int_0^u \frac{ds}{w(t,\ x,\ s)} \tag{37}$$

Since $w(t, x, s) > 0$ when $s < U(t, x)$ and $w = 0$ when $s = U(t, x)$, then $u \to U(t, x)$ as $y \to \infty$ and $0 < u < U(t, x)$ when $0 < y < \infty$, $u|_{y=0} = 0$. Conditions $u|_{t=0} = u_0$ and $u|_{x=0} = u_1$ are also fulfilled by virtue of the conditions $w_0 = u_{0y}$ and $w_1 = u_{1y}$. Function defined by (37) has the derivatives $u_y = w$, $u_{yy} = w_\eta w$, and $u_{yyy} = w_{\eta\eta} u_y^2 + w_\eta u_{yy}$. Derivatives u_t and u_x are given by

$$u_t = -w \int_0^u \frac{w_t(t,\ x,\ s)\,ds}{w^2(t,\ x,\ s)}, \quad u_x = -w \int_0^u \frac{w_x(t,\ x,\ s)\,ds}{w^2(t,\ x,\ s)}$$

Let us put

$$v = \frac{-u_t - uu_x - p_x + \nu u_{yy}}{u_y} \tag{38}$$

We shall show that u and v given by (37) and (38), satisfy the system (1). Differentiating $u_y = w$, we obtain

$$u_{yx} = w_\xi + u_x w_\eta, \quad u_{yt} = w_\tau + u_t w_\eta$$

consequently v possesses a derivative with respect to y. Differentiation of (38) with respect to y, yields

$$v_y u_y + v u_{yy} = -u_{ty} - uu_{xy} - u_y u_x + \nu u_{yyy}$$

or

$$v_y u_y + u_y u_x + u_{yy}\left[\frac{-u_t - uu_x - p_x + \nu u_{yy}}{u_y}\right] + u_{ty} + uu_{xy} - \nu u_{yyy} = 0 \tag{39}$$

Function w satisfies the equation (5). Substitution of u_y for w in (5), yields

$$\nu u_y^2 \left(\frac{u_y u_{yyy} - u_{yy}^2}{u_y^3}\right) - u_{yt} + u_t \frac{u_{yy}}{u_y} - u\left(u_{yx} - \frac{u_x u_{yy}}{u_y}\right) + p_x \frac{u_{yy}}{u_y} = 0 \tag{40}$$

From (40) and (39) it follows, that $v_y u_y + u_x u_y = 0$, i.e.

$$u_x + v_y = 0 \tag{41}$$

Equations (38) and (41) together form the system (1). We shall show now, that v satisfies the condition $v|_{y=0} = v_0(t, x)$. From condition (7) it follows that

$$v_0 = \left(\frac{\nu w w_\eta - p_x}{w}\right)\bigg|_{\eta=0}$$

while (38) implies

$$v\,\big|_{y=0} = \left(\frac{\nu u_{yy} - p_x}{u_y}\right)\bigg|_{y=0} = \left(\frac{\nu w w_\eta - p_x}{w}\right)\bigg|_{\eta=0} = v_0$$

Uniqueness of solution of the problem (1) to (3) follows from the uniqueness of the solution of (5) to (7). For, suppose that two solutions w' and w'' of the problem (5) to (7) exist. Their difference $V^\circ = w' - w''$ will satisfy

$$\nu\,(w')^2 V_{\eta\eta}^\circ - V_\tau^\circ - \eta V_\xi^\circ + p_x V_\eta^\circ + \nu w_{\eta\eta}''\,(w' + w'')\,V^\circ = 0 \tag{42}$$

in Ω, together with the conditions

$$V^\circ|_{\tau=0} = 0, \quad V^\circ|_{\xi=0} = 0, \quad V^\circ|_{\eta=U(\tau,\,\xi)} = 0, \quad (\nu w' V_\eta{}^\circ - v_0 V^\circ + \nu w_\eta{}'' V^\circ)|_{\eta=0} = 0$$

Consider a function V^1 defined by

$$V^\circ = V^1 e^{\alpha\tau - \beta\eta}$$

where α and β are some positive constants. For V^1 from (42), we have

$$\nu (w')^2 V_{\eta\eta}{}^1 - V_\tau{}^1 - \eta V_\xi{}^1 + [p_x - 2\nu (w')^2 \beta] V_\eta{}^1 + \\ + [\nu w_{\eta\eta}{}''(w' + w'') + \nu (w')^2 \beta^2 - \alpha] V^1 = 0 \tag{43}$$

and the conditions

$$V^1|_{\tau=0} = 0,\ V^1|_{\xi=0} = 0,\ V^1|_{\eta=U(\tau,\,\xi)} = 0,\ (\nu w' V_\eta{}^1 + (\nu w_\eta{}'' - v_0 - \nu\beta w')V^1)|_{\eta=0} = 0 \tag{44}$$

If α and β are chosen sufficiently large, then from (44) and (43) it follows that $|V^1|$ cannot assume its greatest value on the internal points of Ω, nor on its boundary. Consequently $V^1 \equiv 0$ and $w' \equiv w''$ in Ω, which was to be proved.

Another proof of uniqueness of the solution of (5) to (7) is given in [8]. A continuous dependence of the solution w of (5) to (7) on the given functions p, v_0, u_0, and u_1 can be proved in an analogous manner. Behavior of the solution of (5) to (7) and of (1) to (3) as $t \to \infty$ was investigated in [9].

Convergence of finite difference approximations to solutions of Prandtl's system was investigated in [10].

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